

Time dependent human hip joint lubrication for periodic motion with stochastic asymmetric density function

KRZYSZTOF WIERZCHOLSKI*

Technical University of Koszalin, Institute of Technology and Education, Koszalin, Poland.

The present paper is concerned with the calculation of the human hip joint parameters for periodic, stochastic unsteady, motion with asymmetric probability density function for gap height. The asymmetric density function indicates that the stochastic probabilities of gap height decreasing are different in comparison with the probabilities of the gap height increasing. The models of asymmetric density functions are considered on the grounds of experimental observations. Some methods are proposed for calculation of pressure distributions and load carrying capacities for unsteady stochastic conditions in a super thin layer of biological synovial fluid inside the slide bearing gap limited by a spherical bone acetabulum. Numerical calculations are performed in Mathcad 12 Professional Program, by using the method of finite differences. This method assures stability of numerical solutions of partial differential equations and gives proper values of pressure and load carrying capacity forces occurring in human hip joints.

Key words: Hip joint, stochastic effects, asymmetric density function, periodic motion

1. Introduction

The problem of hip joint or endoprosthesis lubrication for unsteady stochastic periodic motion has already been considered by Knoll, Cwanek, Mow, Pawlak [1]–[5] and additionally in the author's papers [6]–[10]. Up to now the random considerations and solution methods have been based on the probability symmetric density functions of gap height changes. For example, the Gauss and pseudo-Gauss probability density functions of gap height changes are considered. Such conditions assumed denote that in each arbitrarily chosen time period, the probabilities of gap height decreasing have the same rank as the probability values of gap height increasing. Random changes of the gap height of the natural normal and pathological human hip joint or gap height between head and acetabulum of endoprosthesis are caused mainly by vibrations in unsteady motion and the roughness of the joint surfaces.

From the many experimental observations [1], [11]–[17] it follows that during the arbitrarily chosen time period of unsteady periodic motion of the hip joint the probabilities of the joint gap height decreasing are not equal to the probabilities of the gap increasing. Therefore, in the present paper, the asymmetric density function for probability gap height changes is taken into account.

Moreover, contrary to the foregoing papers [7], [18], the present paper presents a calculation algorithm which satisfies stability conditions of numerous numerical solutions of periodic lubrication problems in the form of partial differential equations and gives real values of fluid velocity components and load carrying capacities occurring in human hip joints and finally real values of friction forces during the forward and backward locomotion of human limbs.

The present research aimed at the following:

- To show a new unified calculation algorithm for stochastic periodic load carrying capacity determi-

* Corresponding author: Krzysztof Wierzecholski, Technical University of Koszalin, Institute of Technology and Education, ul. Śniadeckich 2, 75-453 Koszalin, Poland. Tel: +48 94 3478344, fax: +48 94 3426753, e-mail: krzysztof.wierzecholski@wp.pl

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nation of spherical, bio-bearing human hip joint surfaces in the case of asymmetric density function.

- To indicate the time depended hydrodynamic pressure and load carrying capacity changes caused by the periodic lubrication for various standard deviations with asymmetric probability density function.
- To show the application of the semi-numerical method for stochastic periodic pressure calculation implemented by Mathcad 12 Professional Program.

2. Materials and methods

2.1. Measurements of asymmetric probability density function for gap height

The probability density function of the gap height was assumed by virtue of experimental measurements of cartilage sample roughness and its standard deviations. Random changes of cartilage surface are described using the probability asymmetrical density functions on the basis of comparison between the results of Cwanek and Wierzcholski's experiments [1] and investigations of Dowson and Mow [2], [11], see Fig. 1a, Fig. 1b.

The measurements of samples of natural hip joint surfaces are performed by means of a mechanical or laser sensor, where normal (non-used) and pathological (used) cartilage samples are taken into account. During the measurements performed by means of the mechanical sensor the samples of cartilages having dimensions $2\text{ mm} \times 2\text{ mm}$ are used, and for the measurements by using laser sensor the samples having dimensions of about $10\text{ mm} \times 10\text{ mm}$ are applied. Real gap height changes depend on the variations of cartilage surface, environmental conditions, age of the joint considered, localization of collagen fibers, phospholipid solid lubricant particles [4].

Measurements of the values of changes on the sample surface ($2\text{ mm} \times 2\text{ mm}$) of a normal cartilage resting on the spherical bone head of human hip joint, see Fig. 1, have been performed by using the micro-sensor laser installed in the Rank-Taylor-Hobson-Talyscan-150 Apparatus, and elaborated by means of the Talymap Expert and Microsoft Excel Computer Program [1], [10]. From 29 measured samples the following parameters have been calculated: differences between values of rises and deeps of bone head surfaces in human hip joint (St), arithmetic mean between values of 5 rises and 5 deeps of bone head surface (Sz), standard deviation of probability density function of roughness distribution of cartilage surface (Sa) [1], [6]. The measured values of St oscillate within the interval from 9.79 to 24.7 micrometers. The

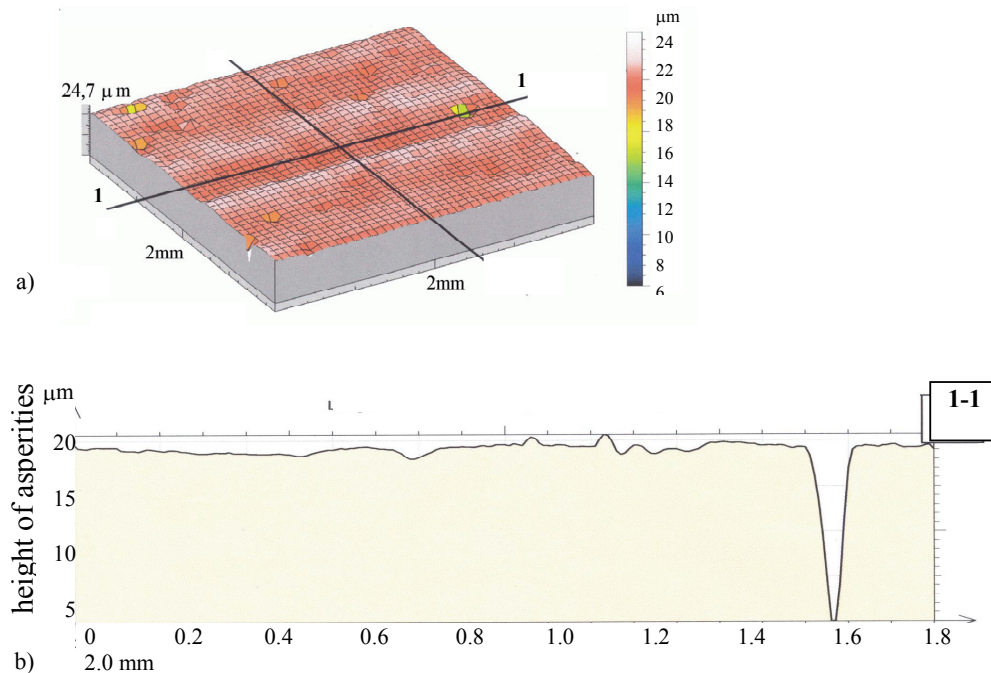


Fig. 1. Random changes of the joint gap height due to roughness and unsteady load: (a) measurement of roughness values on the sample surface ($2\text{ mm} \times 2\text{ mm}$) of normal joint cartilage taken from the bone head of human hip joint, (b) longitudinal section 1-1 of the surface of normal hip joint cartilage of bone head measured by the mechanical sensor

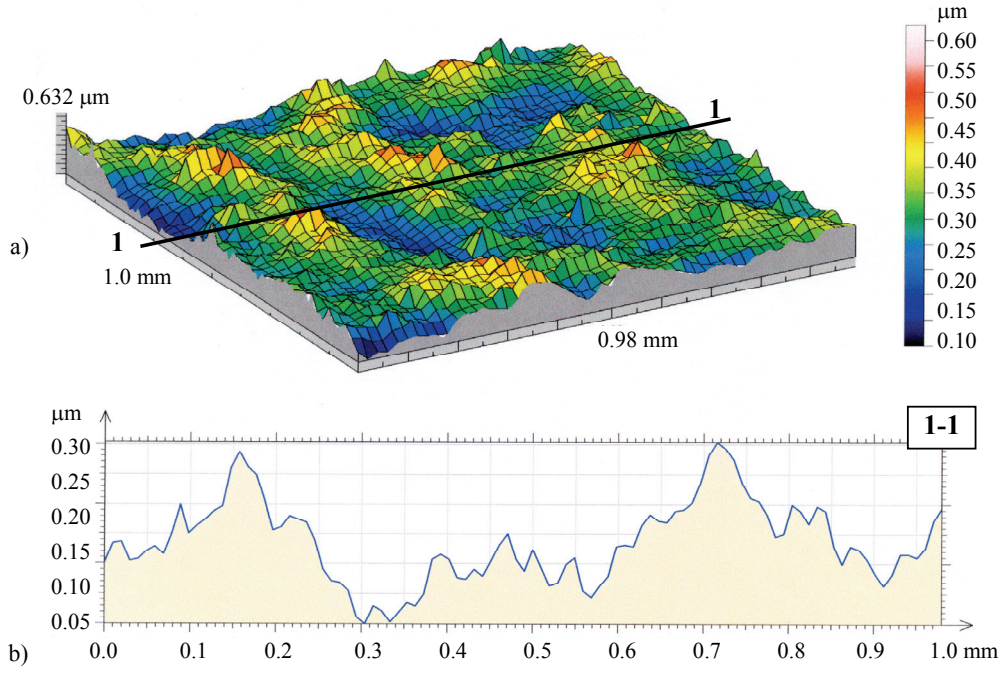


Fig. 2. Sample of a new head of endoprosthesis FRANKOBAL GL46MM: (a) measured roughness of surface, (b) vertical section of surface

measured S_z values fluctuate within the interval from 8.52 to 14.7 micrometers. Finally, we find that the calculated values of S_a are contained in the interval from 0.78 to 1.96 micrometer.

For example, Fig. 2a, Fig. 2b show a hip joint Frankobal's half-endoprosthesis in 3D and 2D geometrical structure where the head is seated onto the pin. Its head consists of the outer metal element having spherical form and the inner element made of polyethylene.

In the geometrical structure presented in Fig. 2 we have the following amplitude parameters: $S_a = 0.0604 \mu\text{m}$, $S_t = 0.632 \mu\text{m}$ and $S_z = 0.538 \mu\text{m}$ [1], [6], [18].

The above-mentioned experimental results obtained complete the thesis about the asymmetric probability density function.

2.2. Random gap height description

The unsteady time dependent gap height has periodic changes. The dimensionless gap height ε_{T1} depends on the variable φ and \mathcal{G}_1 and dimensionless time t_1 and consists of two parts [6], [18], [19]

$$\begin{aligned} \varepsilon_{T1} &= \varepsilon_T / \varepsilon_0 = \varepsilon_{T1s}(\varphi, \mathcal{G}_1, t_1) + \delta_1(\varphi, \mathcal{G}_1, \xi), \\ \varepsilon_{T1s} &= \varepsilon_1 + \varepsilon_{1t}, \end{aligned} \quad (1)$$

where ε_{T1s} denotes the total dimensionless height of nominally smooth part of the thin fluid layer. This part of the gap height contains dimensionless corrections of the gap height caused by the hyper elastic cartilage deformations and is the sum of time independent ε_1 and time dependent ε_{1t} components. The symbol δ_1 denotes the dimensionless random part of changes of the gap height resulting from the vibrations, unsteady loading and surface roughness asperities of cartilage or artificial surfaces measured from the nominal mean level. The symbol ξ describes the random variable, which characterizes the roughness arrangement.

Figure 3 presents the periodic time-dependent gap height with periodic perturbations.

The time-independent value of the smooth part of the gap height has the following dimensional form [1], [18]

$$\begin{aligned} \varepsilon(\varphi, \mathcal{G}_1) &= \varepsilon_0 \varepsilon_1(\varphi, \mathcal{G}_1) \\ &\equiv \Delta \varepsilon_x \cos \varphi \sin \mathcal{G}_1 + \Delta \varepsilon_y \sin \varphi \sin \mathcal{G}_1 - \Delta \varepsilon_z \cos \mathcal{G}_1 - R \\ &\quad + [(\Delta \varepsilon_x \cos \varphi \sin \mathcal{G}_1 + \Delta \varepsilon_y \sin \varphi \sin \mathcal{G}_1 - \Delta \varepsilon_z \cos \mathcal{G}_1)^2 \\ &\quad + (R + \varepsilon_{\min})(R + 2D + \varepsilon_{\min})]^{0.5}, \end{aligned} \quad (2)$$

where ε – characteristic dimensional value of gap height. We assume the centre of spherical bone head to be at the point $O(0, 0, 0)$ and the centre of spherical acetabulum at the point $O_1(x - \Delta \varepsilon_x, y - \Delta \varepsilon_y, z + \Delta \varepsilon_z)$. The eccentricity has the value D (see Fig. 3).

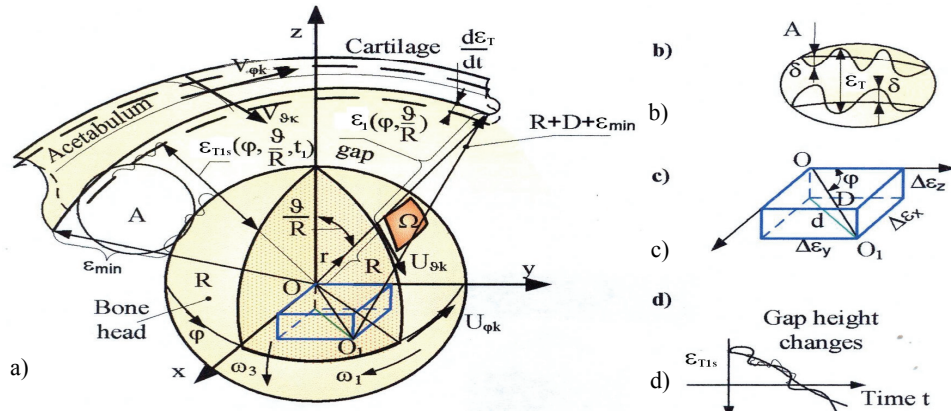


Fig. 3. The gap height time-changes of human hip joint:
 (a) time-changes of the gap and limit velocity values, (b) stochastic changes of the joint gap,
 (c) the centre of the spherical bone head and acetabulum, (d) gap height changes with time

3. Results

3.1. Probability gap height density function and its cumulative values

At first we define the stochastic parameters.

Stochastic changes of the joint gap height are determined by virtue of stochastic parameters such as standard deviation, average standard deviation, expectancy value and distributive function. The parameters mentioned are now defined. Expectancy stochastic operator E imposed on the product of gap height (*) and probability density function f denotes average random changes of gap height defined by [19], [20]

$$E(*) = \int_{-\infty}^{+\infty} (*) \times f(\delta_1) d\delta_1. \quad (3)$$

Standard deviation σ , average standard deviation φ and distributive function F have the following form

$$\sigma \equiv \sqrt{E(*)^2 - E^2(*)}, \quad (4)$$

$\varphi \equiv \sigma / \text{average gap height.}$

$$F(\delta_1) = \int_{-\infty}^{\delta_1} f(\delta_1) d\delta_1. \quad (5)$$

Pseudo-Gaussian symmetrical probability density function

After the first part of measurements of the joint cartilage deformations we observe the case where

probabilities of the gap height random changes decreasing are the same as probabilities of the gap height random variations increasing. This result enables us to create a probability symmetrical density function in the pseudo-Gaussian and triangle form.

- The pseudo-Gaussian symmetrical probability density function has the form [6], [19]

$$f_{g13}(\delta_1) \equiv \begin{cases} \left[1 - \left(\frac{32\delta_1}{35} \right)^2 \right]^3 & \text{for } |\delta_1| \leq \frac{35}{32} = 1.09375, \\ 0 & \text{for } |\delta_1| > 1.09375 = c_{13}, \end{cases}$$

$$\sigma_{g13} = 0.364583. \quad (6)$$

Symbol $c_{13} = 1.093755$ denotes the half total range of random variable of the thin layer thickness for normal hip joint. The symbol $\sigma_{g13} = 0.364583$ denotes the dimensionless standard deviation. To obtain dimensional value of the standard deviation σ_g we must multiply σ_{g13} by the characteristic value of gap height $\varepsilon_0 = 10 \cdot 10^{-6}$ m. In this case, the dimensional standard deviation equals 3.7 micrometers. From the measurements we have obtained the standard deviation value of about 3.5 micrometers for normal cartilage. The various Gaussian dimensionless functions f_{g1} are presented in Fig. 4.

- The classical triangle symmetrical probability density function has the form

$$f_{\Delta}(\delta_1) \equiv \begin{cases} 1 + \delta_1 & \text{for } -1 \leq \delta_1 \leq 0, \\ 1 - \delta_1 & \text{for } 0 \leq \delta_1 \leq 1, \\ 0 & \text{for } |\delta_1| > 1. \end{cases} \quad (7)$$

Triangle function f_{Δ} described by formula (7) is presented in Fig. 5a in graphical form.

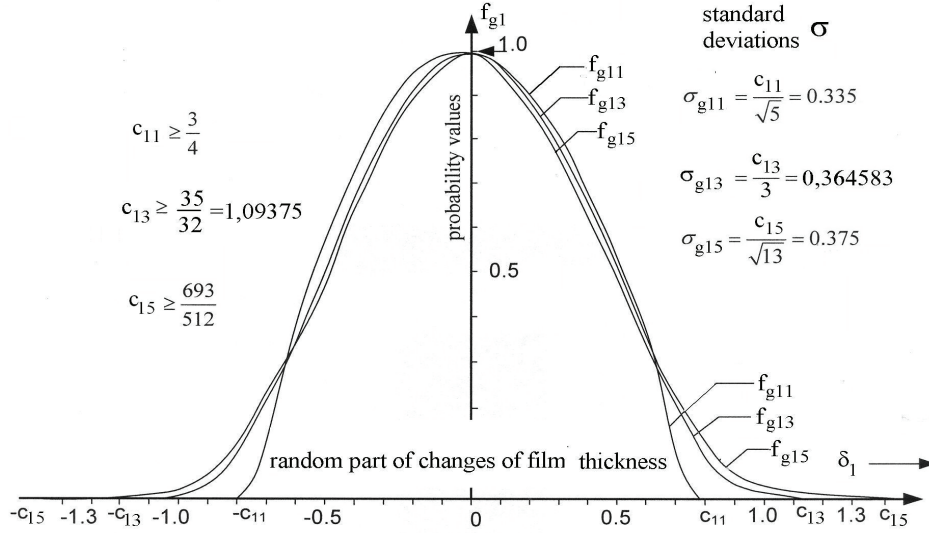


Fig. 4. Symmetric Gaussian probability density dimensionless functions for values of gap height changes if probabilities of gap height decreases and increases have the same order

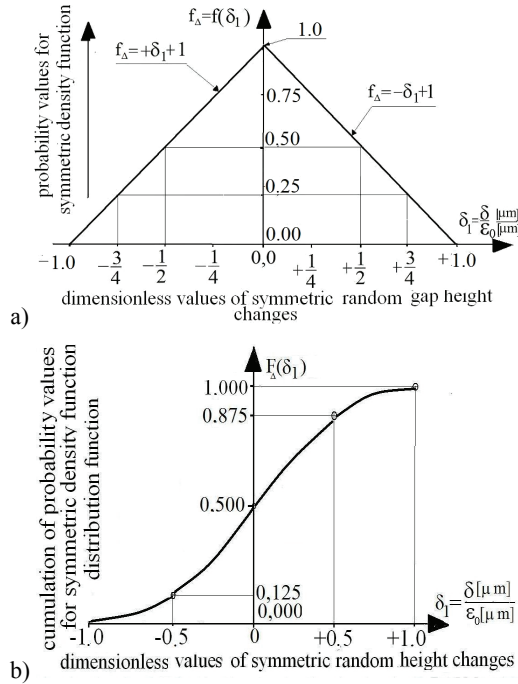


Fig. 5. Probability values of the gap height changes if probability of the gap height decreasing has the same rank as probabilities of the gap height increasing:
 (a) symmetric probability density function,
 (b) cumulative function

We put equation function f_A from (7) into formula (5). Hence distributive function F_A for symmetric triangle density function is presented in Fig. 5b.

We put equation (7) into formula (4). Hence standard deviation for symmetric triangle density function has the following form [19], [20]

$$\sigma_{\Delta 1} = \sqrt{E(f_A)^2 - E^2(f_A)} = 0.408248. \quad (8)$$

To obtain dimensional value of the standard deviation σ_{Δ} we must multiply $\sigma_{\Delta 1}$ by the characteristic value of the gap height $\varepsilon_0 = 10 \cdot 10^{-6}$ m. In this case, the dimensional standard deviation equals 4.08 micrometers for normal cartilage.

3.2. Asymmetrical probability density function for larger increase

From the next part of measurements of joint cartilage deformations we observe the case where probabilities of the gap height random changes increasing are larger than probabilities of the gap height random variations decreasing. The experimental results obtained enable us to create the probability symmetrical density function in the following form

$$f_A(\delta_1) \equiv \begin{cases} \frac{1}{3}\delta_1 + \frac{1}{3} & \text{for } -1 \leq \delta_1 \leq -\frac{1}{4}, \\ 3\delta_1 + 1 & \text{for } -\frac{1}{4} \leq \delta_1 \leq 0, \\ -\frac{1}{3}\delta_1 + 1 & \text{for } 0 \leq \delta_1 \leq \frac{3}{4}, \\ -3\delta_1 + 3 & \text{for } \frac{3}{4} \leq \delta_1 \leq +1, \\ 0 & \text{for } |\delta_1| > 1. \end{cases} \quad (9)$$

Asymmetric function f_A described by formula (9) is presented in Fig. 6 in graphical form.

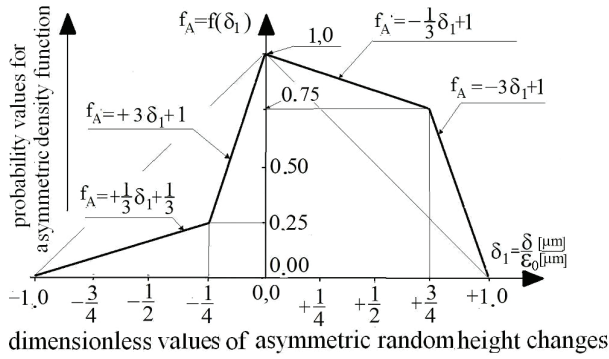


Fig. 6. Asymmetric density function in the case where probabilities of the gap height decreasing are less than probabilities of the gap height increasing

We put equation (9) into formula (5). Hence distributive function for asymmetric density function is described by the following formula (10) and is presented in Fig. 7

$$F_A(\delta_1) = \int_{-\infty}^{\delta_1} f_A(\delta_1) d\delta_1 \quad (10)$$

$$\equiv \begin{cases} \frac{1}{6}\delta_1^2 + \frac{1}{3}\delta_1 + \frac{1}{6} & \text{for } -1 \leq \delta_1 \leq -\frac{1}{4}, \\ \frac{3}{2}\delta_1^2 + \delta_1 + \frac{1}{4} & \text{for } -\frac{1}{4} \leq \delta_1 \leq 0, \\ -\frac{1}{6}\delta_1^2 + \delta_1 + \frac{1}{4} & \text{for } 0 \leq \delta_1 \leq \frac{3}{4}, \\ -\frac{3}{2}\delta_1^2 + 3\delta_1 - \frac{1}{2} & \text{for } \frac{3}{4} \leq \delta_1 \leq +1. \end{cases}$$

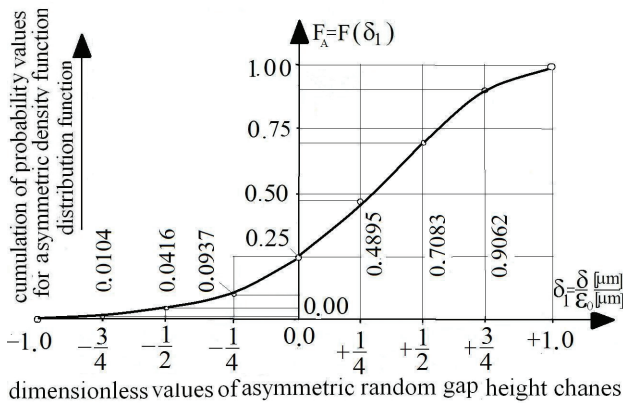


Fig. 7. Distribution function for asymmetrical density function elaborated by virtue of probability values presented in Fig. 6

We put equation (9) into formula (4). Hence standard deviation for asymmetric density function has the following form [19], [20]

$$\sigma_{A1} = \sqrt{E(f_A)^2 - E^2(f_A)} = 0.381837. \quad (11)$$

To obtain dimensional value of the standard deviation σ_A we must multiply σ_{A1} by the characteristic value of gap height $\varepsilon_0 = 10 \cdot 10^{-6}$ m. In this case, the dimensional standard deviation equals 3.81837 micrometers. From the measurements we have obtained the standard deviation value of about 4.0 micrometers for normal cartilage.

3.3. Asymmetrical probability density function for smaller increase

The third part of measurements of joint cartilage deformations shows the case where the probabilities of the gap height random changes increasing are less than the probability of the gap height random variations decreasing. The experimental results obtained enable us to create the probability symmetrical density function in the following form

$$f_a(\delta_1) \equiv \begin{cases} 3\delta_1 + 3 & \text{for } -1 \leq \delta_1 \leq -\frac{3}{4}, \\ \frac{1}{3}\delta_1 + 1 & \text{for } -\frac{3}{4} \leq \delta_1 \leq 0, \\ -3\delta_1 + 1 & \text{for } 0 \leq \delta_1 \leq \frac{1}{4}, \\ -\frac{1}{3}\delta_1 + \frac{1}{3} & \text{for } \frac{1}{4} \leq \delta_1 \leq +1, \\ 0 & \text{for } |\delta_1| > 1. \end{cases} \quad (12)$$

Asymmetric function f_a described by formula (12) is presented in Fig. 8 in graphical form.

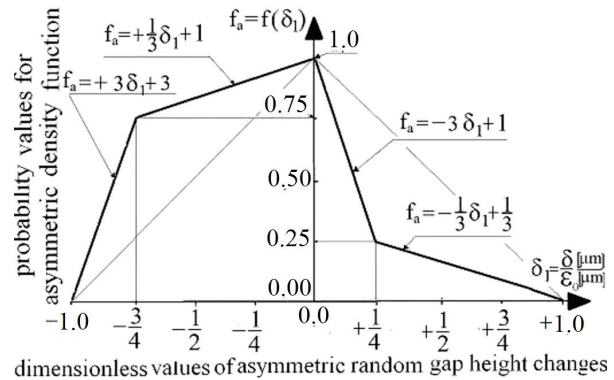


Fig. 8. Asymmetric density function in the case where probabilities of gap height decreasing are significantly larger than probabilities of gap height increasing

We put equation (12) into formula (5). Hence distributive function for asymmetric density function is

described by the following formula (13) and is presented in Fig. 9

$$F_a(\delta_1) = \int_{-\infty}^{\delta_1} f_a(\delta_1) d\delta_1$$

$$\equiv \begin{cases} \frac{3}{2}\delta_1^2 + 3\delta_1 + \frac{3}{2} & \text{for } -1 \leq \delta_1 \leq -\frac{3}{4}, \\ \frac{1}{6}\delta_1^2 + \delta_1 + 3 & \text{for } -\frac{3}{4} \leq \delta_1 \leq 0, \\ -\frac{3}{2}\delta_1^2 + \delta_1 + \frac{3}{4} & \text{for } 0 \leq \delta_1 \leq \frac{1}{4}, \\ -\frac{1}{6}\delta_1^2 + \frac{1}{3}\delta_1 + \frac{5}{6} & \text{for } \frac{1}{4} \leq \delta_1 \leq +1. \end{cases} \quad (13)$$

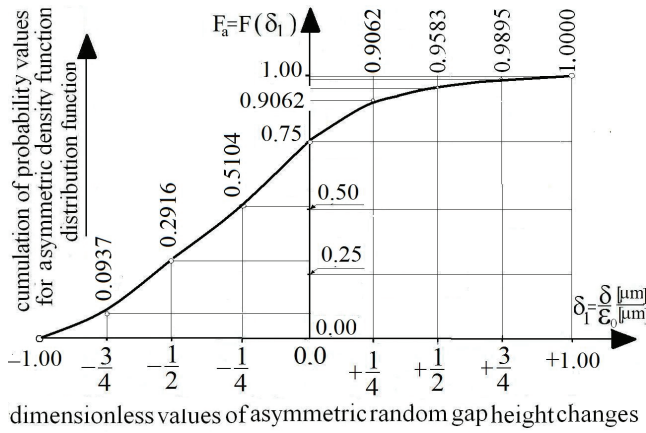


Fig. 9. Distribution function for asymmetrical density function elaborated by virtue of probability values presented in Fig. 8

We put equation (12) into formula (4). Hence standard deviation for asymmetric density function has the following form [19], [20]

$$\sigma_{a1} = \sqrt{E(f_a)^2 - E^2(f_a)} = 0.381837. \quad (14)$$

To obtain dimensional value of the standard deviation σ_a we must multiply σ_{a1} by the characteristic value of gap height $\varepsilon_1 = 10 \cdot 10^{-6}$ m. In this case, the dimensional standard deviation equals 3.81837 micrometers. From the measurements we have obtained the standard deviation value of about 4.0 micrometers for normal cartilage.

3.4. Basic equations and method of solutions

The synovial fluid velocity vector in spherical coordinates has the following components: v_φ , v_r , v_ϑ in φ -, r -, ϑ -directions, respectively. Spherical bone head

moves in the circumferential direction φ and meridian direction ϑ contrarily to the acetabulum. Acetabulum and bone head surfaces vibrate in φ and ϑ directions with various amplitudes and frequencies. Additionally, the acetabulum vibrates in the direction of the gap height which changes with time. The motion and vibrations of joint surfaces cause the synovial fluid flow in the hip joint gap. We assume rotational motion of human bone head in the circumferential direction with the angular velocity ω_1 , and the meridian direction with the angular velocity ω_3 . We have an unsymmetrical unsteady non isothermal synovial fluid flow in the gap, viscoelastic and unsteady properties of synovial fluid. Centrifugal forces are neglected. We denote: the peripheral velocity $U = \omega_1 R$, constant value of the dimensional synovial fluid density $\rho \equiv \rho_0$, changeable synovial fluid viscosity $\eta = \eta_0 \eta_1$, characteristic dimensional value of dynamic viscosity η_0 , time dependent dimensional total gap height ε_T of hip joint, R – radius of bone head, t_0 – characteristic dimensional time value, ε_0 – characteristic dimensional value of gap height. It is assumed that the product of the Deborah, $D_\beta \equiv \beta \omega_1 / \eta_0$, and Strouhal, $Str = R / U t_0$, numbers, i.e. $D_\beta Str$ and the product of the Reynolds, $Re = \rho U \varepsilon_0 / \eta_0$, number, dimensionless clearance ψ , and Strouhal number, i.e., $Re \psi Str$ have the same order of magnitude and that $D_\beta Str \gg D_\beta$. We perform an estimation of basic equation for boundary thin layer flow in a spherical joint gap. Neglecting the terms of radial clearance $\psi \equiv \varepsilon_0 / R \approx 10^{-3}$, and centrifugal forces, in the governing equations in the spherical coordinates: φ , r , ϑ , and taking into account the above mentioned assumptions, we have [10]

$$\frac{\partial v_\varphi}{\partial t} = -\frac{1}{\rho R \sin\left(\frac{\vartheta}{R}\right)} \frac{\partial p}{\partial \varphi} + \frac{\eta_0}{\rho} \frac{\partial}{\partial r} \left(\eta_1 \frac{\partial v_\varphi}{\partial r} \right) + \frac{\beta}{\rho} \frac{\partial^3 v_\varphi}{\partial t \partial r^2} + O_\varphi(D_\alpha), \quad (15.1)$$

$$\frac{\partial p}{\partial r} = (\alpha + 2\beta) \frac{\partial}{\partial r} \left[\left(\frac{\partial v_\varphi}{\partial r} \right)^2 + \left(\frac{\partial v_\vartheta}{\partial r} \right)^2 \right], \quad (15.2)$$

$$\frac{\partial v_\vartheta}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial \vartheta} + \frac{\eta_0}{\rho} \frac{\partial}{\partial r} \left(\eta_1 \frac{\partial v_\vartheta}{\partial r} \right) + \frac{\beta}{\rho} \frac{\partial^3 v_\vartheta}{\partial t \partial r^2} + O_\vartheta(D_\alpha), \quad (15.3)$$

$$\frac{\partial v_\varphi}{\partial \varphi} + R \sin\left(\frac{\vartheta}{R}\right) \frac{\partial v_r}{\partial r} + \frac{\partial}{\partial \vartheta} \left[R v_\vartheta \sin\left(\frac{\vartheta}{R}\right) \right] = 0, \quad (15.4)$$

where $0 < \varphi \leq 2\pi\theta_1$, $0 < \theta_1 < 1$, $b_m \equiv \pi R/8 \leq \vartheta \leq \pi R/2 \equiv b_s$, $0 \leq r \leq \varepsilon_T$, $\varepsilon_T = \varepsilon_0 \varepsilon_{T1}$, ε_{T1} – total dimensionless gap height, $t = t_0 t_1$, t – dimensional time, t_1 – dimensionless time, O_φ , O_ϑ – terms describing the one order smaller viscoelastic oil influences on the flow.

The terms multiplied by the pseudo-viscosity factor β in Pas^2 describe influences of viscoelastic properties of the synovial fluid on the lubrication effects. The convection terms have been neglected. Only time derivatives of velocity component have been retained in the left hand sides of equations (15.1) and (15.3). The joint bone head with hip joint gap and the range of the lubrication region of Ω : $0 \leq \varphi \leq \pi$, $\pi R/8 \leq \vartheta \leq \pi R/2$, are presented in Fig. 10.

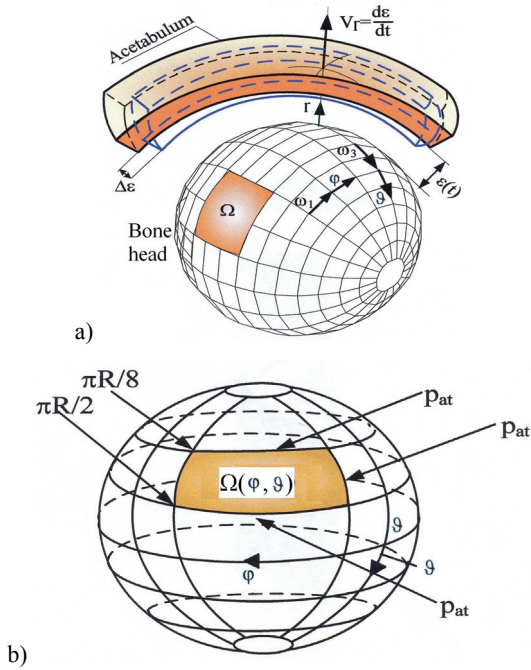


Fig. 10. Hip joint geometry: (a) bone head, hip joint gap and acetabulum, (b) the range of the lubrication region resting on the spherical bone head

It is visible from the measurements performed that radial velocity component has zero value on the bone head and on the acetabulum surface attains a value of the first derivative of the total gap height with respect to time. Acetabulum and bone head move in circumferential, φ , and meridian, ϑ , direction. Moreover, we take into account tangential time-dependent acceleration of bone head surface.

3.5. Stochastic Reynolds equations

Now we show the general form of stochastic Reynolds equation.

Imposing the boundary condition (19.1) on the synovial fluid velocity component (18.2) in radial, i.e., gap height direction we obtain the following non-homogeneous stochastic partial differential Reynolds equation of second order determining the unknown function of hydrodynamic pressure [6], [10]

$$\begin{aligned} & \frac{1}{R} \operatorname{cosec}\left(\frac{\vartheta}{R}\right) \frac{\partial}{\partial \varphi} \left\{ \frac{\varepsilon_0^3 E(\varepsilon_{T1}^3)}{\eta_0 \eta_1} \frac{\partial E(p)}{\partial \varphi} \right\} \\ & + R \frac{\partial}{\partial \vartheta} \left\{ \frac{\varepsilon_0^3 E(\varepsilon_{T1}^3)}{\eta_0 \eta_1} \frac{\partial E(p)}{\partial \vartheta} \sin \frac{\vartheta}{R} \right\} \\ & = 6\omega_1 R \sin\left(\frac{\vartheta}{R}\right) \frac{\varepsilon_0 \partial E(\varepsilon_{T1})}{\partial \varphi} \\ & + 6\omega_3 R^2 \sin(\varphi) \frac{\partial}{\partial \vartheta} \left[\varepsilon_0 E(\varepsilon_{T1}) \sin\left(\frac{\vartheta}{R}\right) \right] \\ & - R \sin\left(\frac{\vartheta}{R}\right) \frac{\varepsilon_0 \partial E(\varepsilon_{T1})}{\partial t} \\ & + O_{pD}(D_\alpha, D_\beta) + O_p(U, V, \omega_{10}, \omega_{30}), \end{aligned} \quad (16)$$

where symbol $E(\varepsilon_{T1})$ denotes expectancy operator of dimensionless gap height.

The values of pressure distribution p are determined in the lubrication region Ω . Total pressure has values of the atmospheric pressure p_{at} on the boundary of the region Ω which is indicated in Fig. 10, and defined – on the basis of medical information – by the following inequalities: Ω : $0 \leq \varphi \leq \pi$, $\pi R/8 \leq \vartheta \leq \pi R/2$. It is a section of the bowl of the sphere. Terms O_{pD} , O_p describe the pressure corrections caused by viscoelastic properties and periodic motion with various frequencies and amplitudes [6], [21].

At first, we determine the stochastic coefficients for the pseudo-Gaussian symmetric probability density function.

By virtue of equations (3) and (6) we have [19], [20], [21]

$$\begin{aligned} E(\varepsilon_{T1}) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1) \\ & \times f_g(\delta_1) d\delta_1 = \varepsilon_{T1s}, \\ E(\varepsilon_{T1}^3) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1)^3 \\ & \times f_g(\delta_1) d\delta_1 = \varepsilon_{T1s}^3 + 3\sigma_{g1}^2 \varepsilon_{T1s}, \end{aligned} \quad (17)$$

where standard deviation equals $\sigma_{g1} = 0.364583$.

Dimensionless average deviation for the Gaussian probability density function is defined in the form: $\beta_{g1} = \sigma_{g1}/\text{avg} \varepsilon_{T1s} < 1/3$ where $\text{avg} \varepsilon_{T1s}$ denotes average dimensionless gap height and $\beta_{g1} > 0$.

Now, we derive the stochastic coefficients for classical triangle symmetrical probability density function.

By virtue of equations (3) and (7) we have [19], [20], [21]

$$\begin{aligned} E(\varepsilon_{T1}) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1) \\ &\quad \times f_{\Delta}(\delta_1) d\delta_1 = \varepsilon_{T1s}, \\ E(\varepsilon_{T1}^3) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1)^3 \\ &\quad \times f_{\Delta}(\delta_1) d\delta_1 = \varepsilon_{T1s}^3 + 3\sigma_{\Delta1}^2 \varepsilon_{T1s}, \end{aligned} \quad (18)$$

where standard deviation equals $\sigma_{\Delta1} = 0.408248$.

Dimensionless average deviation for triangle random function is defined in the form: $\rho_{\Delta1} = \sigma_{\Delta1}/\text{avg}\varepsilon_{T1s}$ where $\rho_{\Delta1} > \rho_{g1}$.

Now, we examine the stochastic coefficients for asymmetrical density function for larger increase.

By virtue of equations (3) and (9) we have [19], [20], [21]

$$\begin{aligned} E(\varepsilon_{T1}) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1) \times f_A(\delta_1) d\delta_1 = \varepsilon_{T1s} + 0.25, \\ E(\varepsilon_{T1}^3) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1)^3 \times f_A(\delta_1) d\delta_1 \\ &\quad (\varepsilon_{T1s} + 0.25)^3 + 3\sigma_{\Delta1}^2 (\varepsilon_{T1s} + 0.25), \end{aligned} \quad (19)$$

where $\sigma_{\Delta1} = 0.3818813068$.

Dimensionless average deviation for random asymmetric function f_A is defined in the form: $\rho_{A1} = \sigma_{\Delta1}/\text{avg}(\varepsilon_{T1s} + 0.25)$, where $\rho_{g1} > \rho_{A1} > 0$.

Now, we conclude with the stochastic coefficients for asymmetrical density function for smaller increase.

By virtue of equations (3) and (12) we have [19], [20], [21]

$$\begin{aligned} E(\varepsilon_{T1}) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1) \times f_a(\delta_1) d\delta_1 = \varepsilon_{T1s} - 0.25, \\ E(\varepsilon_{T1}^3) &= \int_{-\infty}^{+\infty} (\varepsilon_{T1s} + \delta_1)^3 \times f_a(\delta_1) d\delta_1 \\ &\quad = +(\varepsilon_{T1s} - 0.25)^3 + 3\sigma_{a1}^2 (\varepsilon_{T1s} - 0.25), \end{aligned} \quad (20)$$

where $\sigma_{a1} = 0.3818813068$.

Dimensionless average deviation for random asymmetric function f_a is defined in the form: $\rho_{a1} = \sigma_{a1}/\text{avg}(\varepsilon_{T1s} - 0.25)$, where $\rho_{a1} > \rho_{\Delta1} > \rho_{g1} > \rho_{A1} > 0$.

3.6. Numerical calculations

The values of pressure distribution p are determined in the lubrication region Ω . Total pressure has values of the atmospheric pressure p_{at} on the boundary of the region Ω which is indicated in Fig. 10, and defined – on the basis of medical information – by the following inequalities: Ω : $0 \leq \varphi \leq \pi$, $\pi R/8 \leq \alpha_3 \equiv \vartheta \leq \pi R/2$. Numerical calculations are performed in Matlab 7.2 and Mathcad 12 Professional Program by virtue of (16) using the finite difference method [22] for the region Ω , and for the following data:

- radius of the bone head in hip joint $R = 0.0265$ [m],
- angular velocity in circumference direction on the bone head surface $\omega_1 = 1.10$ [s⁻¹],
- angular velocity in meridian direction on the bone head surface $\omega_3 = -0.25$ [s⁻¹],
- angular velocity describing the periodical perturbations in synovial fluid $\omega_0 = 400$ [s⁻¹],
- angular velocity circumference perturbations on the bone head surface $\omega_{10} = 0.10$ [s⁻¹],
- angular velocity meridian perturbations on the bone head surface $\omega_{30} = 0.02$ [s⁻¹],
- components of acetabulum centre eccentricities: $\Delta\varepsilon_x = 2.5$ [μm], $\Delta\varepsilon_y = 0.5$ [μm], $\Delta\varepsilon_z = 2.0$ [μm],
- characteristic value of synovial fluid viscosity $\eta_0 = 0.25$ [Pas],
- characteristic value of synovial fluid density $\rho_0 = 1010$ [kg/m³].

The minimum value of gap height equals $\varepsilon_{\min} = 5.8$ μm, the maximum value of gap height equals $\varepsilon_{\max} = 11.50$ μm. The measured roughness of joint surfaces is taken into account. Pressure values are calculated in the following instants within the time periods: $t = 0$ [s], $t = \pi/\omega_0$ [s], $t = 2\pi/\omega_0$ [s], ... In the first place, we perform numerical calculations for synovial fluid without visco-elastic properties, i.e., for $\beta_0 = 0.00000$ Pas² and in the next place we assume viscoelastic properties for $\beta_0 = 0.00070$ Pas². During the calculations, for the acetabulum, there are taken into account the following time-independent dimensional amplitudes of measured vibration in the form of tangential velocity changes $V_{\varphi_0} = 0.001$ m/s, $V_{\vartheta_0} = 0.0005$ m/s. The time-independent dimensionless scale of gap height perturbation changes has the value $B_e = +0.0001$ [10], [18]. The random effects for the pseudo-Gaussian density function $\rho_{g1} = 1/3$ is taken into account.

Numerical calculations of time-varying pressure distributions are performed by virtue of equation (16) for the gap height (1), (2). Their results are presented

in Fig. 11, Fig. 12, Fig. 13 and Fig. 14. The time period of perturbations equals $t = 2\pi/\omega_0$.

Figure 11 shows the time-varying dimensional pressure distribution in human hip joint where stochastic description of cartilage surface roughness is not taken into account, hence $\beta_{g1} = \beta_{\Delta1} = \beta_{A1} = \beta_{a1} = 0$, and viscoelastic properties of the synovial fluid

are neglected for $\beta_0 = 0.0000 \text{ Pas}^2$. For the time instants: $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, we obtain the following values of joint capacity 1080.8 [N], 955.4 [N], 1080.8 [N], respectively.

Figure 12 shows the time-varying dimensional pressure distribution in human hip joint where stochastic description of roughness of cartilage surface is

$$\begin{aligned} R = 0.0265 \text{ [m]}, \quad \eta_0 = 0.25 \text{ [Pas]}, \quad \omega_0 = 400 \text{ [s}^{-1}\text{]}, \quad \omega_1 = 1.10 \text{ [1/s]}, \quad \omega_3 = -0.25 \text{ [s}^{-1}\text{]} \\ \omega_{10} = 0.15 \text{ [s}^{-1}\text{]}, \quad \omega_{30} = -0.02 \text{ [s}^{-1}\text{]}, \quad \beta_0 = \mathbf{0}, \quad \beta_{g1} = \beta_{\Delta1} = \beta_{A1} = \beta_{a1} = \mathbf{0}, \\ \text{Lubrication surface} = 20.38 \text{ [cm}^2\text{]} \end{aligned}$$

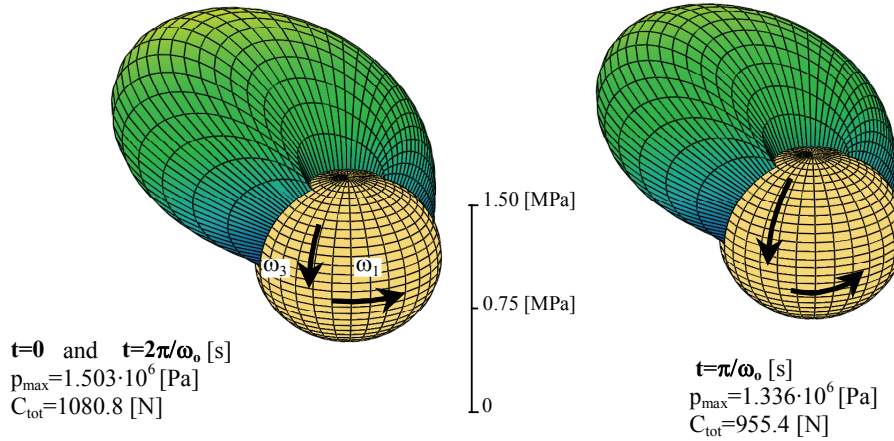


Fig. 11. The dimensional pressure distributions in human hip joint gap, caused by the rotation of the spherical bone head in circumferential direction φ and the meridian direction ϑ , simultaneously, where random effects and visco-elastic properties of synovial fluid are neglected for $\beta_{g1} = \beta_{\Delta1} = \beta_{A1} = \beta_{a1} = 0$, $\beta_0 = 0$. Calculations were performed for the non-zero values of the angular velocity: $\omega_1 = 1.10 \text{ s}^{-1}$, $\omega_3 = -0.25 \text{ s}^{-1}$ and non-zero angular velocity perturbations: ω_{10} , ω_{30} in unsteady flow and for the angular velocity of gap height perturbations, $\omega_0 = 400 \text{ s}^{-1}$

$$\begin{aligned} R = 0.0265 \text{ [m]}, \quad \eta_0 = 0.25 \text{ [Pas]}, \quad \omega_0 = 400 \text{ [s}^{-1}\text{]}, \quad \omega_1 = 1.10 \text{ [1/s]}, \quad \omega_3 = -0.25 \text{ [s}^{-1}\text{]} \\ \omega_{10} = 0.15 \text{ [s}^{-1}\text{]}, \quad \omega_{30} = -0.02 \text{ [s}^{-1}\text{]}, \quad \beta_0 = \mathbf{0.0000} \text{ [Pas}^2\text{]}, \quad \beta_{g1} = 1/3, \\ \text{Lubrication surface} = 20.38 \text{ [cm}^2\text{]} \end{aligned}$$

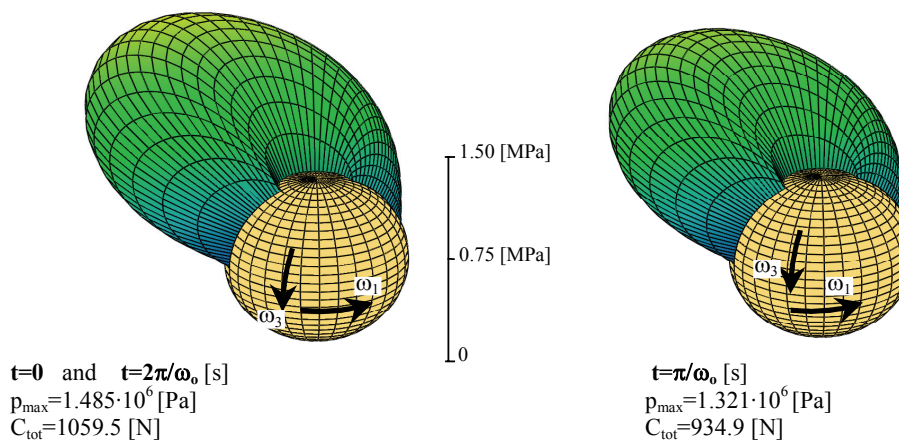


Fig. 12. The dimensional pressure distributions in human hip joint gap, caused by the rotation of the spherical bone head in the circumferential direction φ and the meridian direction ϑ , simultaneously, where only random effects accordingly with the pseudo-Gaussian probability function are taken into account for $\beta_{g1} = 1/3$, and visco-elastic properties of synovial fluid are neglected for $\beta_0 = 0.00000 \text{ Pas}^2$. Calculations were performed for non-zero values of the angular velocity:

$$\omega_1 = 1.10 \text{ s}^{-1}, \quad \omega_3 = -0.25 \text{ s}^{-1} \text{ and the non-zero angular velocity perturbations } \omega_{10}, \omega_{30}, \text{ in unsteady flow and for the angular velocity of gap height perturbations, } \omega_0 = 400 \text{ s}^{-1}$$

$$\begin{aligned}
 R &= 0.0265 \text{ [m]}, \quad \eta_o = 0.25 \text{ [Pas]}, \quad \omega_o = 400 \text{ [s}^{-1}\text{]}, \quad \omega_1 = 1.10 \text{ [1/s]}, \quad \omega_3 = -0.25 \text{ [s}^{-1}\text{]} \\
 \omega_{10} &= 0.15 \text{ [s}^{-1}\text{]}, \quad \omega_{30} = -0.02 \text{ [s}^{-1}\text{]}, \quad \beta_o = 0.0007 \text{ [Pas}^2\text{]}, \quad \rho_{g1} = \rho_{\Delta 1} = \rho_{A1} = \rho_{a1} = 0, \\
 \text{Lubrication surface} &= 20.38 \text{ [cm}^2\text{]}
 \end{aligned}$$

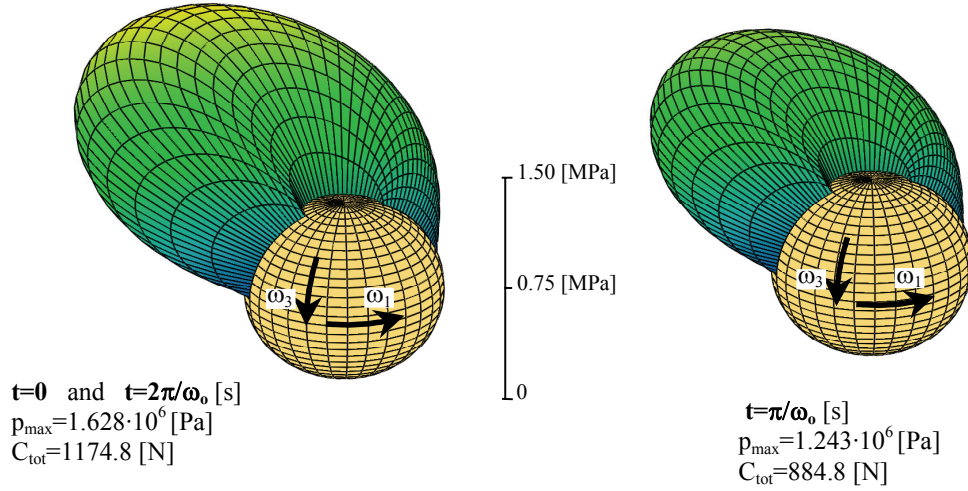


Fig. 13. The dimensional pressure distributions in human hip joint gap, caused by the rotation of the spherical bone head in the circumferential direction φ and the meridian direction ϑ , simultaneously, where all random effects are neglected for $\rho_{g1} = \rho_{A1} = \rho_{a1} = 0$ and visco-elastic properties of synovial fluid taken into account for $\beta_o = 0.00070 \text{ Pas}^2$. Calculations were performed for non-zero values of the angular velocity: $\omega_1 = 1.10 \text{ s}^{-1}$, $\omega_3 = -0.25 \text{ s}^{-1}$ and non-zero angular velocity perturbations ω_{10} , ω_{30} in unsteady flow and for the angular velocity of gap height perturbations, $\omega_o = 400 \text{ s}^{-1}$

$$\begin{aligned}
 R &= 0.0265 \text{ [m]}, \quad \eta_o = 0.25 \text{ [Pas]}, \quad \omega_o = 400 \text{ [s}^{-1}\text{]}, \quad \omega_1 = 1.10 \text{ [1/s]}, \quad \omega_3 = -0.25 \text{ [s}^{-1}\text{]} \\
 \omega_{10} &= 0.15 \text{ [s}^{-1}\text{]}, \quad \omega_{30} = -0.02 \text{ [s}^{-1}\text{]}, \quad \beta_o = 0.0007 \text{ [Pas}^2\text{]}, \quad \rho_{g1} = 1/3, \\
 \text{Lubrication surface} &= 20.38 \text{ [cm}^2\text{]}
 \end{aligned}$$

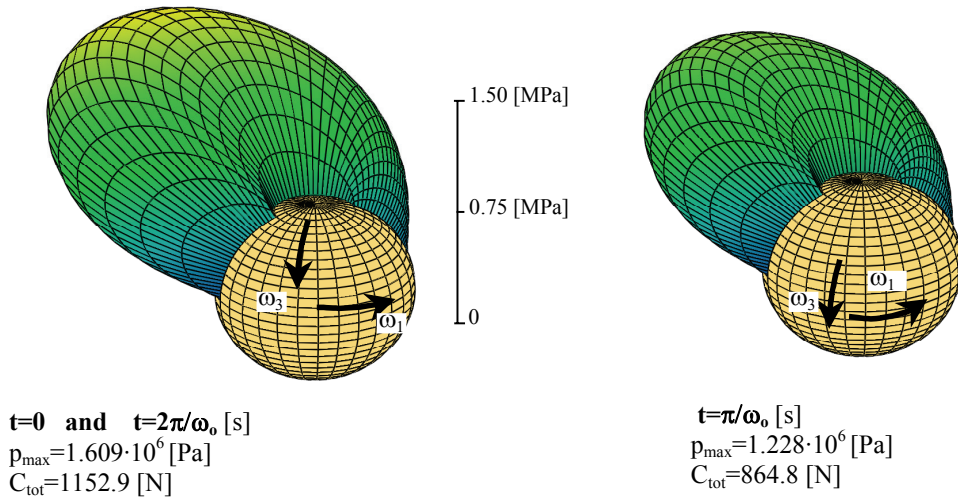


Fig. 14. The dimensional pressure distributions in human hip joint gap, caused by the rotation of the spherical bone head in the circumferential direction φ and the meridian direction ϑ , simultaneously, where only random effects accordingly with the pseudo-Gaussian density function are considered for $\rho_{g1} = 1/3$, and visco-elastic properties of synovial fluid are taken into account for $\beta_o = 0.00070 \text{ Pas}^2$. Calculations were performed for the non-zero values of the angular velocity: $\omega_1 = 1.10 \text{ s}^{-1}$, $\omega_3 = -0.25 \text{ s}^{-1}$ and the nonzero angular velocity perturbations ω_{10} , ω_{30} in unsteady flow and for the angular velocity of gap height perturbations (frequencies), $\omega_o = 400 \text{ s}^{-1}$

taken into account for $\rho_{g1} = 1/3$ and viscoelastic properties of the synovial fluid are neglected, hence for $\beta_o = 0.00000 \text{ Pas}^2$.

For the time instants: $t = 0$, $t = \pi/\omega_o$, $t = 2\pi/\omega_o$, we obtain the following values of joint capacity: 1059.5 [N], 934.9 [N], 1059.5 [N], respectively [6]–[8].

Figure 13 shows the time-varying dimensional pressure distribution in human hip joint where stochastic effects of cartilage surface roughness are neglected, hence $\rho_{g1} = 0$, and viscoelastic properties of the synovial fluid are taken into account for $\beta_0 = 0.00070 \text{ Pas}^2$. For the time instants: $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, we obtain the following values of joint capacity: 1174.8 [N], 884.8 [N], 1174.8 [N], respectively.

Figure 14 shows the time-varying dimensional pressure distribution in the gap of spherical human hip joint for $\rho_{g1} = 1/3$ and $\beta_0 = 0.00070 \text{ Pas}^2$. Stochastic description of cartilage surface roughness is taken into account for $\rho_{g1} = 1/3$. Viscoelastic properties of the synovial fluid are taken into account for $\beta_0 = 0.00070 \text{ Pas}^2$. For the time instants: $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, we obtain the following values of joint capacity: 1152.9 [N], 864.8 [N], 1152.9 [N], respectively [10].

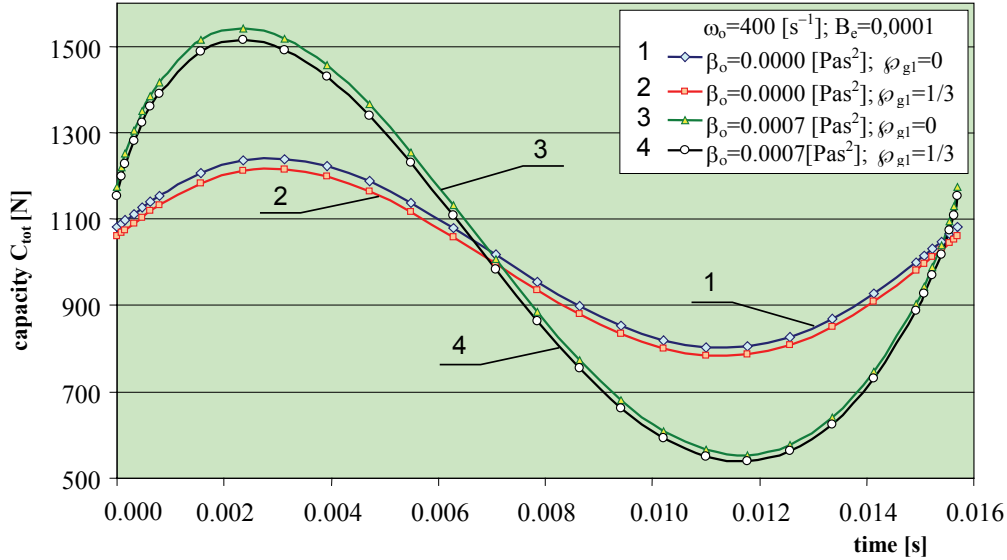


Fig. 15. Dimensional load carrying capacity distributions versus time within the time period where random effects accordingly with the pseudo-Gaussian density function with $\rho_{g1} = 1/3$ are taken into account for four assumptions corresponding to the following four cases presented in Fig. 11, Fig. 12, Fig. 13, Fig. 14, respectively:
 (1). $\rho_{g1} = 0$, $\beta_0 = 0$; (2). $\rho_{g1} = 1/3$, $\beta_0 = 0.0000 \text{ Pas}^2$; (3). $\rho_{g1} = 0$, $\beta_0 = 0.0007$; (4). $\rho_{g1} = 1/3$, $\beta_0 = 0.0007 \text{ Pas}^2$

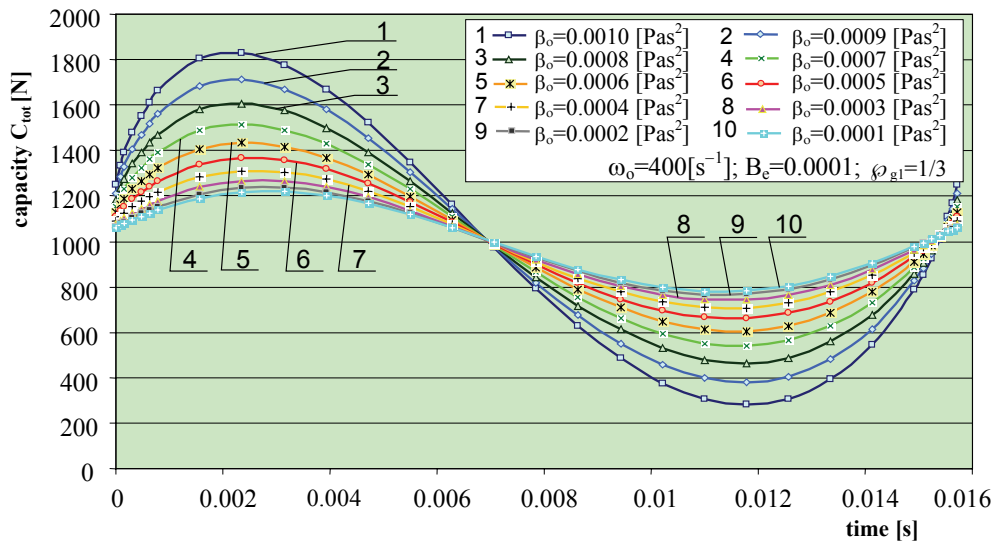


Fig. 16. Load carrying capacity distributions versus time within the time period of perturbations where random effects accordingly with the pseudo-Gaussian density function with $\rho_{g1} = 1/3$ are taken into account for the constant value of periodicity of perturbations $\omega_0 = 400 \text{ s}^{-1}$ and various values of viscoelastic properties of synovial fluid represented by the pseudo-viscosity values inside interval: $0.00001 \text{ Pas}^2 \leq \beta_0 \leq 0.00040 \text{ Pas}^2$

It is easy to see that the pressure distributions and capacities presented in Fig. 11, Fig. 12, Fig. 13 and Fig. 14 for the time instants: $t = 0$ s, $t = 2\pi/\omega_0$ s, i.e., for $t = 2k\pi/\omega_0$ s and $k = 0, 1, 2, \dots$, have the same values. The first upper illustrations on the left sides in Fig. 11, Fig. 12, Fig. 13 and Fig. 14 show the pressure distributions for the origin and end time of the period of perturbations of the motion of human joint. The illustrations for $t = \pi/\omega_0$ in Fig. 11, Fig. 12, Fig. 13 and Fig. 14 show the pressure distributions for the middle-time point of the period of perturbations of the

motion of human joint. Afterwards the pressure distributions return to the distributions which had been shown in the first illustration occurring in each of the figures 11, 12, 13, 14.

- The influences of the random changes of gap height of human joint and visco-elastic synovial fluid properties on the dimensional load carrying capacity distributions in human hip, are presented in Fig. 15 and considered below.

Figure 15 shows the capacity distributions versus time within the time period for constant periodicity of

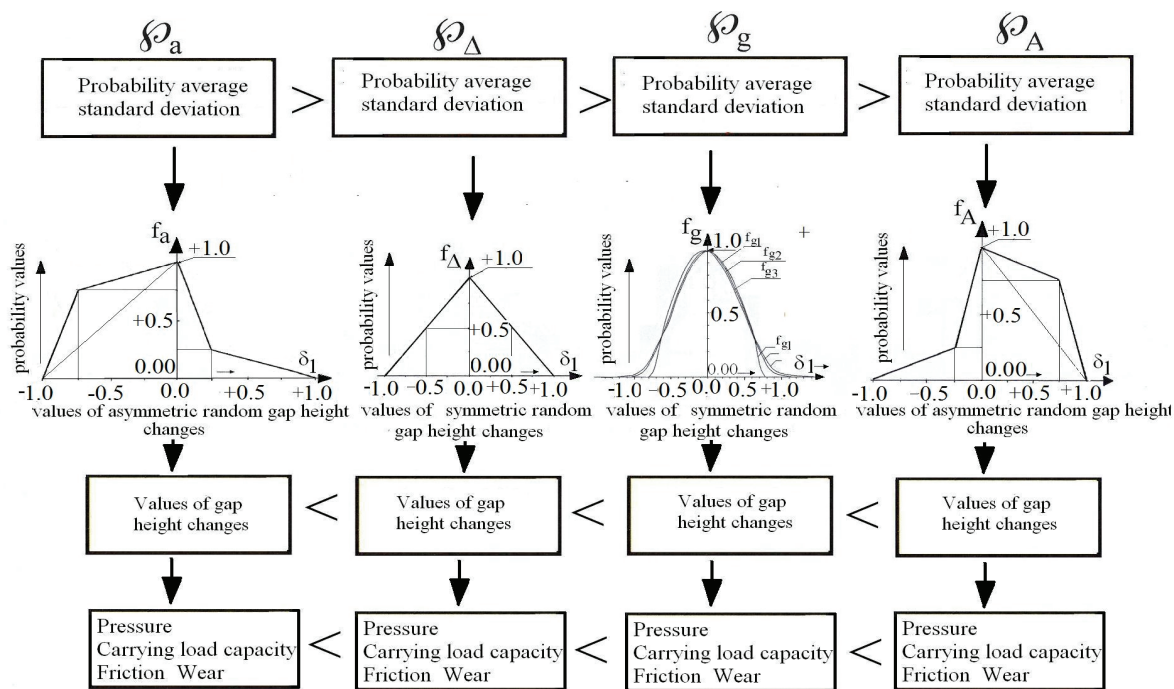


Fig. 17. The influence of the shapes of symmetrical and asymmetrical probability density functions of gap height changes connected with the probability average standard deviation on the hydrodynamic pressure distribution in human hip joint

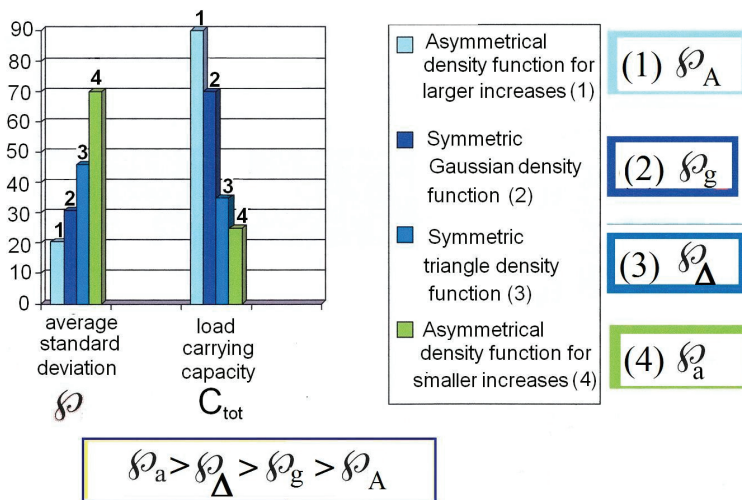


Fig. 18. The graphical illustration of dependences between average standard deviations of human hip joint gap changes and load carrying capacities for unsteady lubrication process

perturbation effects (frequencies), $\omega_0 = 400 \text{ s}^{-1}$, and for four assumptions corresponding to the following four cases presented in Fig. 11, Fig. 12, Fig. 13 and Fig. 14, respectively:

(1) $\rho_{g1} = 0$, $\beta_0 = 0$ (random effects and viscoelastic properties are neglected),

(2) $\rho_{g1} = 1/3$, $\beta_0 = 0$ (random effects are considered, and viscoelastic properties are neglected),

(3) $\rho_{g1} = 0$, $\beta_0 = 0.0007 \text{ Pas}^2$ (random effects are neglected, viscoelastic properties considered),

(4) $\rho_{g1} = 1/3$, $\beta_0 = 0.0007 \text{ Pas}^2$ (random effects and viscoelastic fluid properties are considered).

The symbol $\rho_{g1} = 0$ shows that the random effects are neglected. The case for $\beta_0 = 0$ represents the Newtonian properties of synovial fluid. In the cases mentioned we assume: $\rho_{A1} = \rho_{A1} = \rho_{a1} = 0$, i.e., random effects of triangular and asymmetrical density functions are neglected.

Figure 16 shows the load dimensional carrying capacity distributions versus time within the time period of constant value of its periodicity (frequencies) $\omega_0 = 400 \text{ [s}^{-1}\text{]}$ and for various viscoelastic properties of synovial fluid determined by the following dynamic pseudo-viscosity coefficients: $\beta_0 = 0.0001 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0002 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0003 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0004 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0005 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0006 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0007 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0008 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0009 \text{ [Pas}^2\text{]}$, $\beta_0 = 0.0010 \text{ [Pas}^2\text{]}$.

For unsteady lubrication process occurring in human hip joint, the shapes of probability density functions of the hip joint gap height, have significant influence on hydrodynamic pressure, friction forces and wear. The results obtained after numerous numerical calculations by virtue of equations (16)–(20) are included in illustrations presented in Fig. 17 and Fig. 18.

The above-mentioned results are obtained from the author's own and other authors and co-authors numerous experimental or analytical studies performed under unsteady, stochastic lubrication processes for symmetric and asymmetric density functions occurring in deformed bonehead of human hip joints [1]–[3], [9], [10], [18], [23], [24].

4. Discussion

The equation obtained in this paper for pressure distribution in unsteady periodic conditions, stochastically variable joint gap, visco-elastic properties of synovial fluid, and various frequencies and amplitudes of vibrations of the bone head and acetabulum, tends – in

particular case – to the well known form of Reynolds equation for steady motion, contained in the previous results. From the numerical calculations it follows that viscoelastic properties of synovial fluid increase human hip joint capacities by at least 15%, and – in some cases – even by 60%. From numerical calculations it follows that stochastic description of roughness of bone surfaces and stochastic description of the film thickness of synovial fluid changes the hip joint capacity by about 11%. In this paper, two cases are assumed of probability density functions describing the random part of gap height changes.

We take into account the symmetric probability density function, where random decreases of the gap height changes are the same as the probabilities of increases. Such functions comprise pseudo-Gaussian and triangle function. The results obtained are valid not only for human hip joint but also for other human joints such as foot joint, elbow joint, knee joint, shoulder joint. On the grounds of experiments we present an asymmetric probability density function, where probabilities of the gap height increases are larger (smaller) than probabilities of the gap height random decreases.

To compare the pressure distribution we define average deviation coefficient ρ as a ratio of the standard deviation and dimensionless average gap height.

If average standard deviation is equal zero, then the stochastic influences are neglected.

If average standard deviation increases, then the joint load carrying capacity decreases.

It was found that for the pseudo-Gaussian probability density function, the average standard deviation coefficient attains approximately the value 1/3.

For asymmetric probability density function the average standard deviation coefficient is smaller and larger than 1/3 and has always positive values.

Initial research indicates that:

- For unsteady lubricant conditions, if average standard deviation increases (decreases), vibration amplitude, i.e., maximum value of the gap height change decreases (increases). Hence the pressure decreases (increases) and friction force and wear decreases (increases). These phenomena are illustrated in Fig. 17 and Fig. 18.

- For steady lubricant conditions, average standard deviation increases (decreases) imply increases (decreases) of the gap height changes. Hence the pressure increases (decreases) and friction force and wear increases (decreases).

In the author's opinion, the real human hip joint lubrication is always unsteady with random effects and almost in each case the probability density func-

tion of the values of joint gap height, load carrying capacity, and friction forces is asymmetric.

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