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Effective moduli of trabecular bone

JÓZEF JOACHIM TELEGA, ANDRZEJ GAŁKA, STANISŁAW TOKARZEWSKI

Institute of Fundamental Technological Research, 00-049 Warsaw, Poland

Cancellous bones with plate-like and rod-like architecture are modelled as linear elastic cellular solid with regular microstructure. General formulae for the effective moduli are derived. Specific examples show plate-like and rod-like cancellous bones with isotropic trabeculae.

Keywords: trabecular bone, cellular bone, effective elastic moduli, homogenisation

1. Introduction

Bones occur in two forms: as a dense solid (*compact bone*) and as a porous network of connecting rods and plates (*cancellous or trabecular bone*). The most obvious difference between these two types of bones is in their relative densities measured by volume fraction of solids. Bone with a volume fraction less than 70% is classified as cancellous, while that with a volume fraction over 70% is compact [1]. Most bones in the body consist of a dense compact bone forming an outer shell surrounding a core of spongy cancellous bone. This paper deals with cancellous bones of volume fractions of solid less than 30%, cf. also [4].

The aim of this paper is to develop a macroscopic model of cancellous bones by using homogenisation methods. Both plate-like and rod-like structures of bones are investigated. Our approach enables us to model anisotropic behaviour of bone.

2. Effective moduli of trabecular bone with plate-like architecture

Let Ω denote a bounded open subset of $\mathbf{R}^3$. Y denotes the basic cell, cf. [2, 3]. Y^* denotes the part of Y occupied by the material. It is assumed that the hole T in Y does not intersect the boundary ∂Y, though this assumption may be weakened. By Ω_ε^* we denote the part of Ω occupied by the material. Here $\varepsilon > 0$ is a small parameter. We assume that the holes do not meet the boundary $\partial\Omega$.

Let us consider the following boundary value problem of linear elasticity [2]:

$$\frac{\partial}{\partial x_j}\left(C_{ijkl}\left(\frac{\mathbf{x}}{\varepsilon}\right)\frac{\partial u_k^\varepsilon}{\partial x_l}\right)+f_i=0 \text{ in } \Omega_\varepsilon^*,$$

$$u_k^\varepsilon=0 \text{ on } \partial\Omega, \qquad C_{ijkl}\left(\frac{\mathbf{x}}{\varepsilon}\right)\frac{\partial u_k^\varepsilon}{\partial x_l}n_j=0 \text{ on } \partial\Omega_\varepsilon^*/\partial\Omega, \tag{1}$$

where $\mathbf{n} = (n_j)$ is the unit vector normal to $\partial\Omega_\varepsilon^*/\partial\Omega$. The remaining symbols are standard.

The first stage of homogenisation consists in passing with ε to zero. Afterwards the trabeculae are characterised by a small parameter $\eta > 0$. The second step of homogenisation consists in passing with η to zero. Let now the basic cell Y be given by

$$Y=\left[-\frac{1}{2},\frac{1}{2}\right)\times\left[-\frac{1}{2},\frac{1}{2}\right)\times\left[-\frac{1}{2},\frac{1}{2}\right]. \tag{2}$$

Due to periodicity the homogenised coefficients do not depend on the basic cell and consequently one may take a translated cell of the basic one. Consequently, we take the translated cell represented by Fig. 1a.

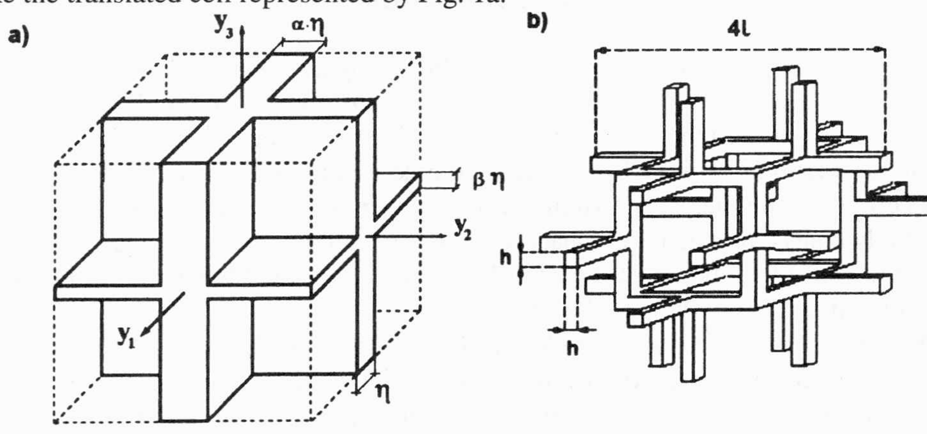

Fig. 1. The models of trabecular bone: a) plate-like architecture, b) rod-like architecture

We observe that the thicknesses of three orthogonal plates are not necessarily equal, thus allowing for macroscopically orthotropic response of the trabecular bone. By letting η to tend to zero we derive the macroscopic moduli [2]:

$$C^*_{ijmn}=(1+\alpha+\beta)C_{ijmn}-C_{ijp1}\left(\mathbf{C}_1^{-1}\right)_{pq}C_{1qmn}-\alpha\, C_{ijp2}\left(\mathbf{C}_2^{-1}\right)_{pq}C_{2qmn}$$
$$-\beta\, C_{ijp3}\left(\mathbf{C}_3^{-1}\right)_{pq}C_{3qmn}. \tag{3}$$

For isotropic and homogeneous trabeculae we get

$$\mathbf{C}^* = \begin{bmatrix} \frac{4\mu(\alpha+\beta)(\lambda+\mu)}{2\mu+\lambda} & \frac{2\beta\lambda\mu}{2\mu+\lambda} & \frac{2\alpha\lambda\mu}{2\mu+\lambda} & 0 & 0 & 0 \\ \frac{2\beta\lambda\mu}{2\mu+\lambda} & \frac{4\mu(1+\beta)(\lambda+\mu)}{2\mu+\lambda} & \frac{2\lambda\mu}{2\mu+\lambda} & 0 & 0 & 0 \\ \frac{2\alpha\lambda\mu}{2\mu+\lambda} & \frac{2\lambda\mu}{2\mu+\lambda} & \frac{4\mu(1+\alpha)(\lambda+\mu)}{2\mu+\lambda} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\alpha\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\beta\mu \end{bmatrix} \quad (4)$$

Obviously, here Voigt's notation has been used.

Having in mind Fig. 1a, the physical effective elasticity tensor, now denoted by $\mathbf{C}^{\mathrm{eff}}$, is given by

$$\mathbf{C}^{\mathrm{eff}} = \frac{v}{1+\alpha+\beta}\mathbf{C}^*, \quad (5)$$

where v is the volume fraction.

Two specific examples are depicted in Figs. 2 and 3.

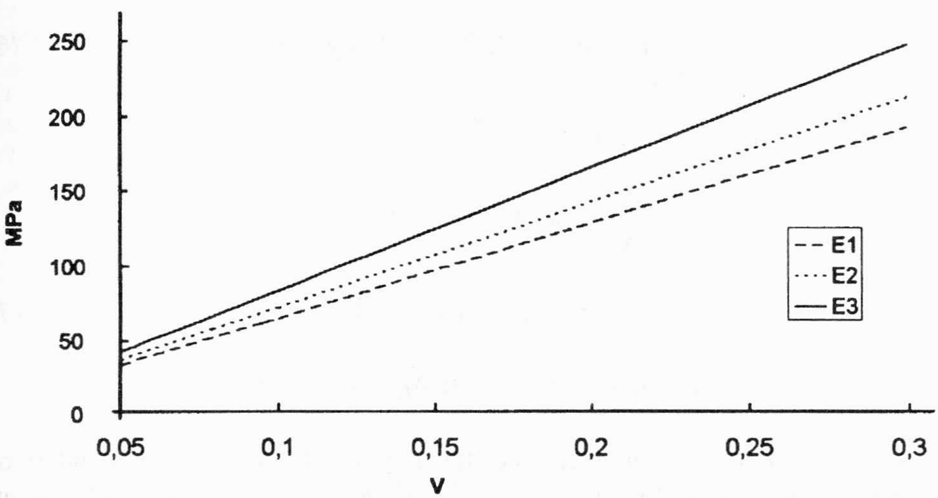

Fig. 2. Young's moduli in the three orthotropic principal directions versus bone volume fractions; $\alpha = 56/67$, $\beta = 73/134$; isotropic trabeculae with $E = 1$ Gpa, $v = 0.35$

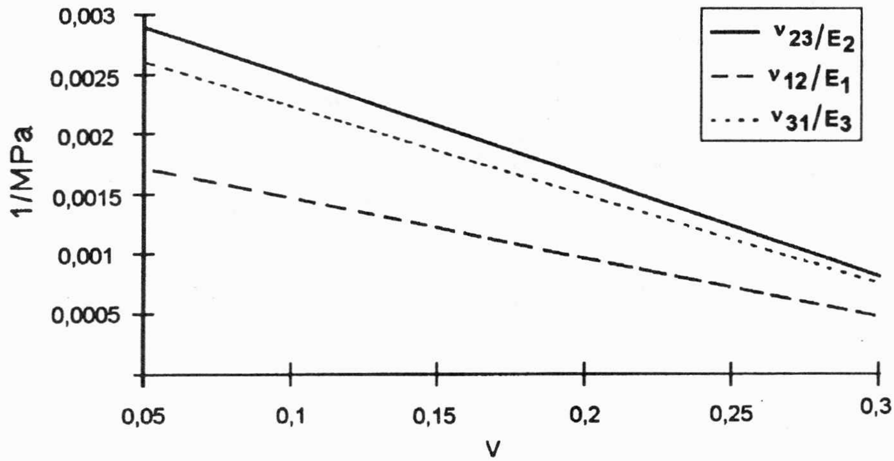

Fig. 3. Poisson's ratio divided by Young's moduli in the three orthotropic principal directions versus bone volume fractions; $\alpha = 56/67$, $\beta = 73/134$; isotropic trabeculae with $E = 1$ Gpa, $\nu = 0.35$

3. Trabecular bone with rod-like architecture

For a periodical network of elastic rods the effective moduli q_{ijrs} are written in the form

$$q_{ijrs} = \frac{1}{|Y|} \int_{Y_t} \Pi_{ij}^{rs} dy, \qquad \Pi_{ij}^{rs} = a_{ijkh} \left(e_{kh} \left(\mathbf{U}^{rs} (\mathbf{y}) \right) \right). \tag{6}$$

The local problem is now given by

$$\begin{gathered} -\frac{\partial}{\partial y} a_{ijkh} \left(e_{kh}^{y} \mathbf{U}^{rs} (\mathbf{y}) \right) = 0 \text{ in } Y_t, \\ e_{kh}^{y} \left(\mathbf{U}^{rs} (\mathbf{y}) \right) \Big|_{\partial Y_+} = e_{kh}^{y} \left(\mathbf{U}^{rs} (\mathbf{y}) \right) \Big|_{\partial Y_-}, \\ \Pi_{ij}^{rs} N_j = 0 \text{ on } \partial Y_t', \qquad \partial Y_t \partial Y_t' \cap \partial Y_+ \cap \partial Y_- . \end{gathered} \tag{7}$$

Here Y_t, ∂Y_t, ∂Y_+, ∂Y_- denote: the region occupied by the elastic rods, the boundary of the region of elastic rods and the opposite walls of the basic cell, respectively. As an example we evaluate the effective elastic moduli of the regular network of elastic rods shown in Fig. 1b. Then the macroscopic elasticity tensor exhibits cubic symmetry and is given by:

$$\mathbf{C}=\begin{bmatrix} \frac{15E_sJ}{2l^4} & \frac{9E_sJ}{4l^4} & \frac{9E_sJ}{4l^4} & 0 & 0 & 0 \\ \frac{9E_sJ}{4l^4} & \frac{15E_sJ}{2l^4} & \frac{9E_sJ}{4l^4} & 0 & 0 & 0 \\ \frac{9E_sJ}{4l^4} & \frac{9E_sJ}{4l^4} & \frac{15E_sJ}{2l^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3E_sJ}{4l^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3E_sJ}{4l^4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3E_sJ}{4l^4} \end{bmatrix} \tag{8}$$

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