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A tribology of curvilinear bone surfaces in human joints

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Two co-operating bone surfaces in human joint are separated by the thin layer of synovial fluid. Considering hydrodynamic lubrication problems in such a case, we should take account of actual various geometry of biobearing bone surfaces. In order to study the synovial fluid flow in biobearing gap, its velocity components and proper Reynolds equations allowing determination of hydrodynamic pressure distributions are derived.

Keywords: biotribology, curvilinear bone

1. Preface

The layer boundary simplifications of basic equations of motion for hydrodynamic symmetrical flow of synovial fluid in biobearing gap were presented by Wierzcholski [1, 2]. Flows in such gaps are not considered in theoretical works because till now the proper Lamé coefficients for realistic co-operating bone surfaces are not derived (the only exception is paper [3]). The flow through a narrow gap depends heavily on the gap geometry. Therefore, the present paper shows the simulation of hydrodynamic, unsymmetrical synovial fluid flow by means of the system of non-linear partial differential equations using the Lamé coefficients which describe the orthogonal curvilinear biobearing surfaces, e.g. parabolic, hyperbolic and spherical.

The main aim of this paper is:

• To find the velocity components of synovial fluid flow and hydrodynamic pressure for parabolic, hyperbolic and spherical bone surfaces with non-monotonic sections in longitudinal direction in biobearing human gap.

• To show a general analytical solution to lubrication problem for hydrodynamic unsymmetrical flow of synovial fluid in curvilinear biobearing gap.

This paper is the continuation of the papers [1–3].

2. Biobearing geometry

The mathematical theory of slide biobearing computations is based on the real model of the synovial flow and real biobearing gap in the thin layer between two co-operating sliding bone surfaces. The solution of the lubrication problem for biobearing depends on joint geometry [6].

Figure 1 presents radial elbow joint with spherical and hyperbolic bone surfaces. Figure 2 shows the human elbow joint with parabolic and hyperbolic bone surfaces, where α_1 is the circumference direction, α_3 is the generating line of rotational bone direction and α_2 denotes the gap height direction.

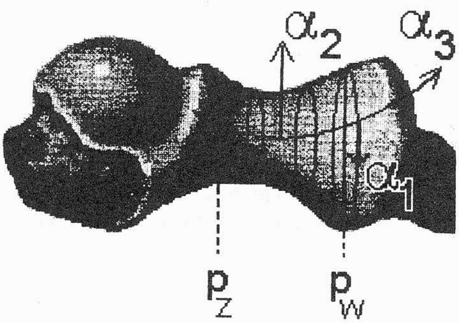

Fig. 1. Radial elbow joint (articulatio radioulnaris proximalis cubiti)

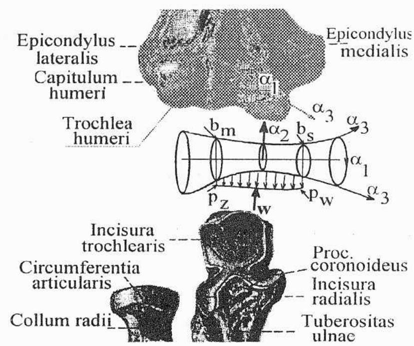

Fig. 2. Elbow joint (articulatio cubiti)

We consider axially unsymmetrical, stationary synovial fluid flow in the film between two rotational generating lines [1]. For thin layer boundary simplifications, equations of conservation of momentum and equation of continuity have the following dimension form [3]:

$$0 = -\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\left(\eta_p \frac{\partial v_1}{\partial \alpha_2}\right), \tag{1}$$

$$0 = \frac{\partial p}{\partial \alpha_2}, \tag{2}$$

$$-\frac{\rho v_1^2}{h_1 h_3}\frac{\partial h_1}{\partial \alpha_3} = -\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3} + \frac{\partial}{\partial \alpha_2}\left(\eta_p \frac{\partial v_3}{\partial \alpha_2}\right), \tag{3}$$

$$h_3 \frac{\partial v_1}{\partial \alpha_1} + h_1 h_3 \frac{\partial v_2}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3}\left(h_1 v_3\right) = 0\,, \tag{4}$$

where circumference direction $0 \le \alpha_1 \le \alpha_\varepsilon$, length direction $b_m \le \alpha_3 \le b_s$, gap height direction $0 \le \alpha_2 \le \varepsilon$. The symbols $b_{m.}$, b_s denote the limits of the synovial fluid flow in the length (generating line) direction, $\alpha_e < 2\pi$ denotes the limit of the flow in the circumference direction. The gap height ε must be a function of the variable α_1, i.e. $\varepsilon_1 = \varepsilon_1(\alpha_1)$ for unsymmetrical flow [2, 3]. The symbol η_p denotes dynamic viscosity of synovial fluid with non-Newtonian properties, and ρ is synovial fluid density. System (1)–(4) describes four unknowns, namely three components of the synovial fluid velocity v_i $(\alpha_1, \alpha_2, \alpha_3)$ for $i = 1, 2, 3$ and the pressure $p = p(\alpha_1, \alpha_2)$. The symbol v_1 denotes the velocity component in the circumference direction, v_2 is the velocity component in the gap height direction and v_3 is the velocity component in the length direction. Symbols $h_1 = h_1(\alpha_3)$ and $h_3 = h_3(\alpha_3)$ are the Lamé coefficients which are dependent on the biobearing geometry. Both Lamé coefficients h_1, h_3 depend on α_3 only because bone surfaces are rotational and non-monotone in the α_3 direction [3].

3. Lamé coefficients

Hyperbolic geometry is shown in Fig. 3. Radius vector of hyperbolic surface has the form:

$$\mathbf{r}_0 = \mathbf{i}\,x_1 + \mathbf{j}\,x_2 + \mathbf{k}\,x_3, \tag{5}$$

where **i**, **j**, **k** are the unit vectors in the Cartesian coordinates x_i. Dependences between the Cartesian x_i and hyperbolic coordinates α_i for $i = 1, 2, 3$ on the hyperbolic surface are as follows [3]:

$$x_1 = \frac{a\cos\alpha_1}{\cos^2(\alpha_3\Lambda)}, \quad x_2 = \frac{a\sin\alpha_1}{\cos^2(\alpha_3\Lambda)}, \quad x_3 = \frac{\tan(\alpha_3\Lambda)}{\Lambda}, \tag{6}$$

whereas

$$\Lambda \equiv \frac{1}{b}\sqrt{\frac{w}{a}}, \; 0 \le \alpha_1 \le 2\pi, \quad 0 \le \alpha_2 \le \varepsilon, \quad |\alpha_3| \le \frac{1}{\Lambda}\arccos\sqrt{\frac{a}{a+w}}.$$

We introduce the following denotations: a – the smallest radius, $a_1 = a + w$ – the largest radius, $w \equiv a_1 - a$, $2b$ – the bearing length. Equations (6) satisfy the following equation of hyperbolic surface:

$$x_1^2 + x_2^2 = \left[a + \left(\frac{x_3}{b}\right)^2 w\right]^2. \tag{7}$$

If we neglect the terms of the order $\psi \approx \varepsilon/a \approx 10^{-3}$, we obtain the Lamé coefficients for hyperbolic bone surface in the following final form [3]:

$$h_1 = \frac{a}{\cos^2(\alpha_3 \Lambda)}, \qquad h_2 = 1, \qquad h_3 = \frac{1}{\cos^2(\alpha_3 \Lambda)}\sqrt{1 + 4(a\Lambda)^2 \tan^2(\alpha_3 \Lambda)}. \tag{8}$$

Parabolic surface is shown in Fig. 4. The radius vector has the form (5), and in this case we have the following dependences between the Cartesian x_i and parabolic coordinates α_i, for $i = 1, 2, 3$, on the parabolic surface [4]:

$$x_1 = a\cos^2(\alpha_3 \Lambda)\cos\alpha_1, \qquad x_2 = a\cos^2(\alpha_3 \Lambda)\sin\alpha_1, \qquad x_3 = \frac{1}{\Lambda}\sin(\alpha_3 \Lambda), \tag{9}$$

whereas

$$0 \le \alpha_1 < \alpha_e < 2\pi, \quad |\alpha_3| \le \frac{1}{\Lambda}\arccos\sqrt{\frac{a-w}{a}}, \qquad \Lambda \equiv \frac{1}{b}\sqrt{\frac{w}{a}},$$

where a is the largest radius, a_1 is the smallest radius, $2b$ is the bearing length, and $w = a - a_1$. Equations (9) satisfy the following equation of parabolic surface:

$$x_1^2 + x_2^2 = \left[a - \left(\frac{x_3}{b}\right)^2 w\right]^2. \tag{10}$$

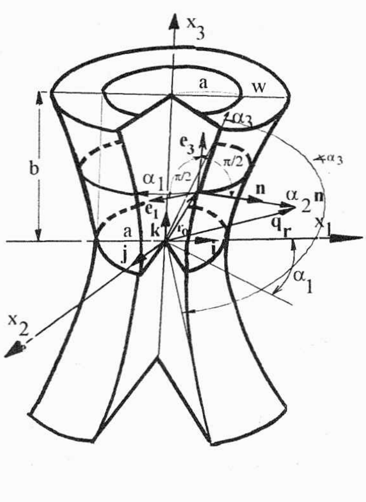

Fig. 3. Hyperbolic geometry

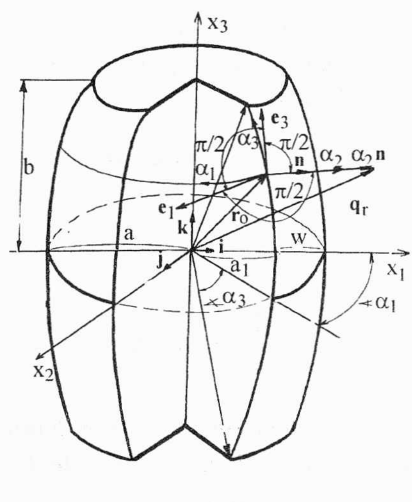

Fig. 4. Parabolic geometry

If we neglect the terms of the order $\psi \approx \varepsilon / a \approx 10^{-3}$, we obtain the Lamé coefficients for parabolic bone surface in the following final form [3]:

$$h_1 = a\cos^2(\alpha_3 \Lambda), \quad h_2 = 1, \quad h_3 = \sqrt{1 + 4a^2\Lambda^2 \sin^2(\alpha_3 \Lambda)}\cos(\alpha_3 \Lambda). \tag{11}$$

The dependences between the Cartesian x_1, x_2, x_3 and spherical coordinates on the sphere with the radius R: $\alpha_1 \equiv \phi$, $\alpha_3/R \equiv \vartheta$ (see Fig. 5) are as follows [4]:

$$x_1 = R\sin\alpha_3\cos\alpha_1, \quad x_2 = R\sin\alpha_3\sin\alpha_1, \quad x_3 = R\cos\alpha_3, \tag{12}$$

where: $0 \le \alpha_1 \le 2\pi$, $0 \le \alpha_2 \le \varepsilon$, $0 \le \alpha_3 \le \pi R$, R being the radius of the sphere. Equations (12) satisfy the following equation of the sphere:

$$x_1^2 + x_2^2 + x_3^2 = R^2. \tag{13}$$

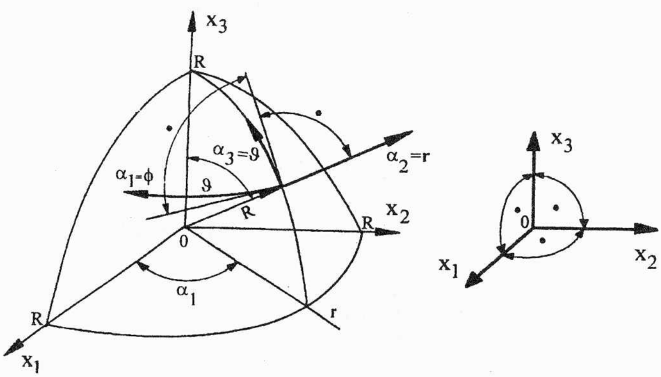

Fig. 5. Spherical geometry

We neglect the terms of the order $\psi \approx \varepsilon/R \approx 10^{-3}$ and we obtain the Lamé coefficients for spherical bone surface in the following final form:

$$h_1 = R\sin\frac{\alpha_3}{R}, \qquad h_2 = 1, \qquad h_3 = 1. \tag{14}$$

4. A sketch of integration method

The synovial flow is generated by the rotation of the bone. Acetabulum is motionless, thus:

$$\begin{aligned} &v_1 = \omega h_1, v_2 = 0, v_3 = 0 \text{ for } \alpha_2 = 0 \text{(bone surface)}; \\ &v_1 = v_2 = v_3 = 0 \text{ for } \alpha_2 = \varepsilon(\alpha_1). \end{aligned} \tag{15}$$

Taking into account the above boundary conditions for v_1, v_3 we obtain from Eqs. (1), (3) the following particular solutions [4]:

$$v_1(\alpha_1,\alpha_2,\alpha_3)=-\int_{\alpha_2}^{\varepsilon(\alpha_1)}N d\alpha_2,\quad v_3(\alpha_1,\alpha_2,\alpha_3)=\int_0^{\alpha_2}\frac{K}{\eta_p}d\alpha_2-\frac{H(\alpha_2)}{H(\alpha_2=\varepsilon)}\int_0^{\varepsilon}\frac{K}{\eta_p}d\alpha_2, \tag{16}$$

where:

$$N(\alpha_1,\alpha_2,\alpha_3)\equiv\frac{1}{\eta_p}\left(C+\frac{\alpha_2}{h_1}\frac{\partial p}{\partial\alpha_1}\right),\qquad H(\alpha_2)\equiv\int_0^{\alpha_2}\frac{1}{\eta_p}d\alpha_2,$$

$$K(\alpha_1,\alpha_2,\alpha_3)\equiv\frac{1}{h_3}\left\{\alpha_2\frac{\partial p}{\partial\alpha_3}-\frac{1}{h_1}\frac{\partial h_1}{\partial\alpha_3}\int_0^{\alpha_2}\rho\left[\int_{\alpha_{22}}^{\varepsilon}N(\alpha_1,\alpha_{23},\alpha_3)d\alpha_{23}\right]^2 d\alpha_{21}\right\}, \tag{17}$$

whereas

$$0\le\alpha_{23}\le\alpha_{22}\le\alpha_{21}\le\alpha_2\le\varepsilon(\alpha_1).$$

The integration constant C satisfies an algebraic equation:

$$-\omega h_1=\int_0^{\varepsilon}N(\alpha_1,\alpha_2,\alpha_3)d\alpha_2. \tag{18}$$

The continuity equation (4) we integrate with respect to the variable α_2. Imposing condition $v_2 = 0$ for $\alpha_2 = 0$ upon the synovial fluid velocity component v_2 we obtain:

$$v_2(\alpha_1,\alpha_2,\alpha_3)=-\frac{1}{\rho h_1}\int_0^{\alpha_2}\left[\frac{\partial}{\partial\alpha_1}(\rho v_1)+\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}(\rho h_1 v_3)\right]d\alpha_2. \tag{19}$$

We substitute the velocity components (16) into formula (19). The boundary condition $v_2 = 0$ for $\alpha_2 = \varepsilon$ imposed upon the velocity component v_2 leads to the following modified Reynolds equation [4]:

$$\frac{\partial}{\partial\alpha_1}\left\{\int_0^{\varepsilon}\left[\rho\int_{\varepsilon}^{\alpha_2}N(\alpha_1,\alpha_{21},\alpha_3)d\alpha_{21}\right]d\alpha_2\right\}+\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left\{\int_0^{\varepsilon}\left[\rho h_1\int_0^{\alpha_2}\frac{1}{\eta_p}K(\alpha_1,\alpha_{21},\alpha_3)\,d\alpha_{21}\right]d\alpha_2\right\}$$

$$=\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left\{\frac{\int_0^{\varepsilon}[\rho h_1(\alpha_3)H(\alpha_2)]\,d\alpha_2}{H(\alpha_2=\varepsilon)}\int_0^{\varepsilon}\frac{1}{\eta_p}K(\alpha_1,\alpha_{21},\alpha_3)d\alpha_{21}\right\}, \tag{20}$$

where $0 \le \alpha_1 < 2\pi$, $b_m \le \alpha_3 \le b_s$, $0 \le \alpha_{21} \le \alpha_2 \le \varepsilon$. This equation determines the pressure function $p[\alpha_1, \alpha_3]$.

5. Final analytical solutions

In hyperbolic curvilinear coordinates the synovial fluid velocity components have the following forms:

$$v_1 = -\frac{\varepsilon^2}{2\eta_o a} s(1-s)\frac{\partial p}{\partial \alpha_1}\cos^2(\Lambda\alpha_3) + \frac{\omega\alpha}{\cos^2(\Lambda\alpha_3)}(1-s), \tag{21}$$

$$v_2 = \frac{1}{6}\frac{\varepsilon^3}{\eta_o a^2} s^2(1-s)\cos^4(\Lambda\alpha_3)$$

$$\times\left[\frac{\partial^2 p}{\partial \alpha_1^2} + \frac{a}{\sqrt{1+4a^2\Lambda^2\tan^2(\Lambda\alpha_3)}}\frac{\partial}{\partial\alpha_3}\left(\frac{a}{\sqrt{1+4a^2\Lambda^2\tan^2(\Lambda\alpha_3)}}\frac{\partial p}{\partial\alpha_3}\right)\right], \tag{22}$$

$$v_3 = -\frac{\varepsilon^2}{2\eta_o} s(1-s)\frac{\partial p}{\partial \alpha_3}\cos^2(\Lambda\alpha_3)\frac{1}{\sqrt{1+4a^2\Lambda^2\tan^2(\Lambda\alpha_3)}}. \tag{23}$$

The pressure function p for hyperbolic gap we find from the following modified Reynolds equation:

$$\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3}{\eta_o}\frac{\partial p}{\partial\alpha_1}\right) + \frac{a}{\sqrt{1+4a^2\Lambda^2\tan^2(\Lambda\alpha_3)}}\frac{\partial}{\partial\alpha_3}\left(\frac{\varepsilon^3}{\eta_o}\frac{a}{\sqrt{1+4a^2\Lambda^2\tan^2(\Lambda\alpha_3)}}\frac{\partial p}{\partial\alpha_3}\right)$$

$$= \frac{6\omega a^2}{\cos^4(\Lambda\alpha_3)}\frac{\partial\varepsilon}{\partial\alpha_3}, \tag{24}$$

whereas

$$s = \frac{\alpha_2}{\varepsilon}, \quad 0 \le \alpha_2 \le \varepsilon, \quad b_m \le \alpha_3 \le b_s, \quad 0 \le \alpha_1 < 2\pi.$$

In parabolic curvilinear coordinates the synovial fluid velocity components have the following form:

$$v_1 = -\frac{\varepsilon^2}{2\eta_o a} s(1-s) \frac{\partial p}{\partial \alpha_1} \frac{1}{\cos^2(\Lambda\alpha_3)} + \omega a(1-s)\cos^2(\Lambda\alpha_3), \tag{25}$$

$$v_2 = \frac{1}{6}\frac{\varepsilon^3}{\eta_o a^2} s^2(1-s) \frac{1}{\cos^4(\Lambda\alpha_3)}$$

$$\times\left[\frac{\partial^2 p}{\partial\alpha_1^2} + \frac{a\cos(\Lambda\alpha_3)}{\sqrt{1+4a^2\Lambda^2\sin^2(\Lambda\alpha_3)}}\frac{\partial}{\partial\alpha_3}\left(\frac{a\cos(\Lambda\alpha_3)}{\sqrt{1+4a^2\Lambda^2\sin^2(\Lambda\alpha_3)}}\frac{\partial p}{\partial\alpha_3}\right)\right], \tag{26}$$

$$v_3 = -\frac{\varepsilon^2}{2\eta_o} s(1-s) \frac{\partial p}{\partial \alpha_3} \frac{1}{\cos(\Lambda\alpha_3)} \frac{1}{\sqrt{1+4a^2\Lambda^2\sin^2(\Lambda\alpha_3)}}. \tag{27}$$

The pressure function p for parabolic gap can be found from the following modified Reynolds equation:

$$\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3}{\eta_o}\frac{\partial p}{\partial\alpha_1}\right) + \frac{a\cos(\Lambda\alpha_3)}{\sqrt{1+4a^2\Lambda^2\sin^2(\Lambda\alpha_3)}}\frac{\partial}{\partial\alpha_3}\left(\frac{\varepsilon^3}{\eta_o}\frac{a\cos(\Lambda\alpha_3)}{\sqrt{1+4a^2\Lambda^2\sin^2(\Lambda\alpha_3)}}\frac{\partial p}{\partial\alpha_3}\right)$$

$$= 6\omega a^2 \frac{\partial\varepsilon}{\partial\alpha_3}\cos^4[\Lambda\alpha_3], \tag{28}$$

whereas

$$s = \frac{\alpha_2}{\varepsilon}, \qquad 0 \le \alpha_2 \le \varepsilon, \qquad b_m \le \alpha_3 \le b_s, \qquad 0 \le \alpha_1 < 2\pi.$$

In spherical curvilinear coordinates the synovial fluid velocity components have the following forms:

$$v_1 = -\frac{\varepsilon^2}{2\eta_o R} s(1-s) \frac{\partial p}{\partial \alpha_1} \frac{1}{\sin\frac{\alpha_3}{R}} + \omega R(1-s)\sin\frac{\alpha_3}{R},$$

$$v_2 = \frac{1}{6}\frac{\varepsilon^3}{\eta_o R^2} s^2(1-s) \frac{1}{\sin^2\left(\frac{\alpha_3}{R}\right)}\left[\frac{\partial^2 p}{\partial\alpha_1^2} + R^2\sin\left(\frac{\alpha_3}{R}\right)\frac{\partial}{\partial\alpha_3}\left(\sin\left(\frac{\alpha_3}{R}\right)\frac{\partial p}{\partial\alpha_3}\right)\right],$$

$$v_3 = -\frac{\varepsilon^2}{2\eta_o} s(1-s)\frac{\partial p}{\partial \alpha_3}.$$

The pressure function p for a spherical gap can be found from the following modified Reynolds equation:

$$\frac{\partial}{\partial \alpha_1}\left(\frac{\varepsilon^3}{\eta_o}\frac{\partial p}{\partial \alpha_1}\right) + R\sin\left(\frac{\alpha_3}{R}\right)\frac{\partial}{\partial \alpha_3}\left[\frac{\varepsilon^3}{\eta_o}R\sin\left(\frac{\alpha_3}{R}\right)\frac{\partial p}{\partial \alpha_3}\right]$$

$$= 6\omega R^2 \sin^2\left(\frac{\alpha_3}{R}\right)\frac{\partial \varepsilon}{\partial \alpha_3},$$

whereas

$$s = \frac{\alpha_2}{\varepsilon}, \qquad 0 \le \alpha_2 \le \varepsilon, \qquad 0 \le \alpha_3 \le \pi R, \qquad 0 \le \alpha_1 < 2\pi.$$

6. Conclusions

Determination of the Lamé coefficients (8),(11),(14) for the hyperbolic, parabolic and spherical rotational bone surfaces enables us to obtain the analytical solutions of unsymmetrical hydrodynamic lubrication problem for biobearing of human elbow joint which is shown in Fig. 1 and Fig. 3 and which is described by means of the system of non-linear partial differential equations (1)–(4) of the second order.

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