

# **Axisymmetrical flow of synovial fluid between curvilinear bone surfaces in human joints**

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Bone and cartilage co-operate in human joint. The gap between their surfaces is filled with synovial fluid. In this case, the solution of the lubrication problem should be based on actual, various geometry of co-operating surfaces of human joint. In this paper we consider a symmetrical flow of synovial fluid between spherical, hyperbolic and parabolic bone surfaces in human joint. In order to study the symmetrical flow of synovial fluid in joint gap, we obtain velocity components and hydrodynamic pressure distributions in final, dimensionless form.

*Key words: final solutions of symmetrical flow*

## **1. Preface**

The layer boundary simplifications and solutions of basic equations of motion for hydrodynamic, unsymmetrical synovial fluid flow simulation by means of the system of non-linear partial differential equations using the Lamé coefficients are presented in [1], [2]. The Lamé coefficients describe the orthogonal, curvilinear biobearing surfaces, e.g. parabolic, hyperbolic and spherical. Solution of this problem was not formulated in its final form, since no analytical solution of the modified Reynolds equation was possible. Various joints are presented in figure 1.

The layer boundary simplifications of basic equations of motion for the hydrodynamic symmetrical flow of synovial fluid in joint gap were presented by WIERZCHOLSKI in [3], [4], but only in a general way in arbitrary curvilinear coordinates for curvilinear co-operating surfaces. The final form of symmetrical flow of synovial fluid in spherical hyperbolic and parabolic coordinates has not been considered to date in theoretical papers, because no proper Lamé coefficients for realistic co-operating bone surfaces have been derived (except that derived by WIERZCHOLSKI [3]). The flow in a narrow gap depends heavily on the gap geometry.

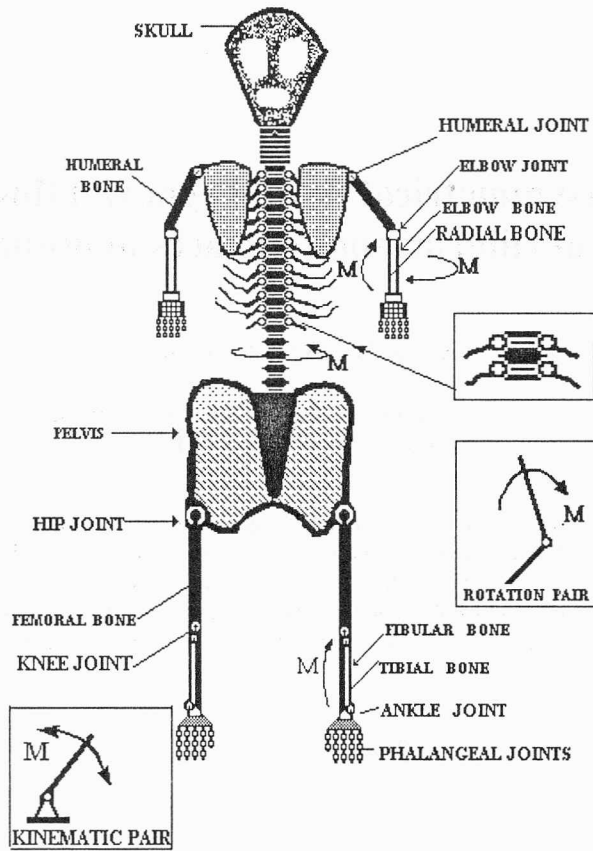


Fig. 1. Draft of human skeleton with joints

The main aim of this paper was to find the final form of the velocity components  $v_i(\alpha_2, \alpha_3)$  of synovial fluid symmetrical flow and hydrodynamic pressure  $p(\alpha_3)$  for spherical, parabolic and hyperbolic bone surfaces with non-monotonic sections in the longitudinal direction in human joint gap, for variable gap height in longitudinal direction, i.e.,  $\varepsilon(\alpha_3)$ , and for variable viscosity of synovial fluid. This paper is a continuation of [3], [4].

## 2. Axisymmetrical flow in a narrow gap between two rotational surfaces

The mathematical theory of the computation of flow in human joint is based on a real model of synovial flow and real joint gap in a thin layer between two cooperating sliding surfaces.

Solution of the lubrication problem of human joint depends on this joint geometry.

We introduce the following, curvilinear coordinates:  $\alpha_1$  – circumference direction,  $\alpha_3$  – generating path of rotational head of the bone,  $\alpha_2$  – gap height direction.

We consider axisymmetrical, stationary flow of synovial fluid in the film between two surfaces [3]. For thin layer boundary simplifications, equations of conservation of momentum and continuity equation have the following dimensional form [3]:

$$0 = \frac{\partial}{\partial \alpha_2} \left( \eta_p \frac{\partial v_1}{\partial \alpha_2} \right), \quad (1)$$

$$0 = \frac{\partial p}{\partial \alpha_2}, \quad (2)$$

$$-\frac{\rho v_1^2}{h_1 h_3} \frac{\partial h_1}{\partial \alpha_3} = -\frac{1}{h_3} \frac{\partial p}{\partial \alpha_3} + \frac{\partial}{\partial \alpha_2} \left[ \eta_p \left( \frac{\partial v_3}{\partial \alpha_2} \right) \right], \quad (3)$$

$$0 = h_1 h_3 \frac{\partial v_2}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3} (h_1 v_3), \quad (4)$$

where  $0 \leq \alpha_1 \leq 2\pi$ ,  $0 \leq \alpha_2 \leq \varepsilon$ ,  $b_m \leq \alpha_3 \leq b_s$ , whereas  $b_m, b_s$  are the limits of lubrication in the direction  $\alpha_3$ . The term on the left-hand side of equation (3) denotes centrifugal acceleration.

In this case, the gap height may be a function of the variable  $\alpha_3$  only, i.e.  $\varepsilon = \varepsilon_0 \varepsilon_1(\alpha_3)$ , or may be constant. Moreover, if a generating line of rotational surface is monotonic for  $\alpha_3 \in [b_m, b_s]$ , we have  $h_3 \equiv 1$  in equations (1)–(4).

For the axisymmetrical flow of synovial fluid, three components of synovial fluid velocity depend on the variables  $\alpha_3$  and  $\alpha_2$  only, i.e.  $v_i = v_i(\alpha_2, \alpha_3)$  for  $i = 1, 2, 3$ . Pressure function depends on  $\alpha_3$  only, i.e.  $p = p_0 p_1(\alpha_3)$ .

### 3. Lamé coefficients

• The dependencies between the Cartesian  $x_1, x_2, x_3$  and spherical coordinates on the sphere with the radius  $R_0$ :  $\alpha_1 \equiv \phi$ ,  $\alpha_3/R_0 \equiv \vartheta = \chi_1$  (see figure 2) are as follows [4]:

$$x_1 = R_0 \sin \frac{\alpha_3}{R} \cos \alpha_1, \quad x_2 = R_0 \sin \frac{\alpha_3}{R} \sin \alpha_1, \quad x_3 = R_0 \cos \frac{\alpha_3}{R}, \quad (5)$$

where  $0 \leq \alpha_1 \leq 2\pi$ ,  $0 \leq \alpha_2 \leq \varepsilon$ ,  $0 \leq \alpha_3 \leq \pi R_0$ . Equation (5) satisfies the following equation of the sphere:

$$x_1^2 + x_2^2 + x_3^2 = R_0^2. \quad (6)$$

We neglect the terms of the order  $\psi = \varepsilon/R_0 \approx 10^{-3}$  and we obtain the Lamé coefficients for spherical bone surface in the final following form:

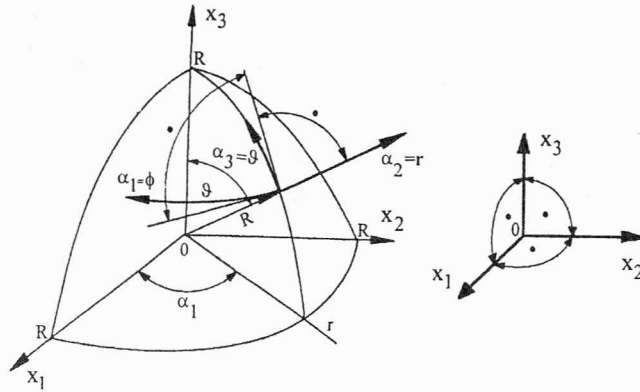


Fig. 2. Spherical geometry

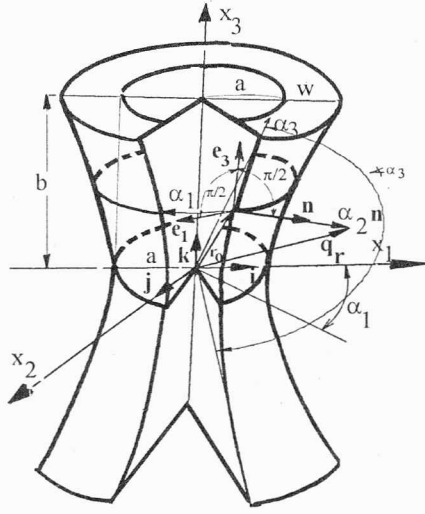


Fig. 3. Hyperbolic geometry

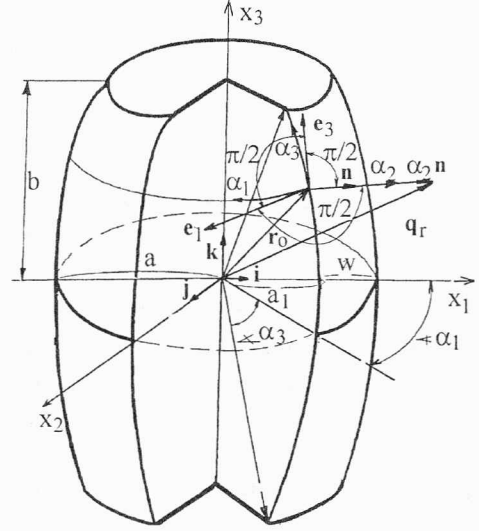


Fig. 4. Parabolic geometry

$$h_1 = R_0 \sin \frac{\alpha_3}{R_0}, \quad h_2 = 1, \quad h_3 = 1. \quad (7)$$

• Hyperbolic geometry is shown in figure 3. Radius vector of hyperbolic surface has the form:

$$\mathbf{r}_0 = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3, \quad (8)$$

where  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  are unit vectors in the Cartesian coordinates  $x_i$ . The dependencies between the Cartesian  $x_i$  and hyperbolic coordinates  $\alpha_i$  for  $i = 1, 2, 3$  on the hyperbolic surfaces are as follows [3]:

$$x_1 = \frac{a \cos \alpha_1}{\cos^2(\alpha_3 \Lambda)}, \quad x_2 = \frac{a \sin \alpha_1}{\cos^2(\alpha_3 \Lambda)}, \quad x_3 = \frac{\tan(\alpha_3 \Lambda)}{\Lambda}, \quad (9)$$

whereas

$$\Lambda \equiv \frac{1}{b} \sqrt{\frac{w}{a}}, \quad 0 \leq \alpha_1 \leq 2\pi, \quad 0 \leq \alpha_2 \leq \varepsilon, \quad |\alpha_3| \leq \frac{1}{\Lambda} \arccos \sqrt{a/(a+w)}.$$

We make the following notations:  $a$  – the smallest radius,  $a_1 = a + w$  – the largest radius,  $w \equiv a_1 - a$ ,  $2b$  – the bearing length. Equation (9) satisfies the following equation of hyperbolic surface:

$$x_1^2 + x_2^2 = \left[ a + \left( \frac{x_3}{b} \right)^2 w \right]^2. \quad (10)$$

Neglecting the terms of the order  $\psi = \varepsilon/a \approx 10^{-3}$ , we obtain the values of the Lamé coefficients for a hyperbolic bone surface in the following final form:

$$h_1 = \frac{a}{\cos^2(\alpha_3 \Lambda)}, \quad h_2 = 1, \quad h_3 = \frac{1}{\cos^2(\alpha_3 \Lambda)} \sqrt{1 + 4(a\Lambda)^2 \tan^2(\alpha_3 \Lambda)}. \quad (11)$$

• Parabolic surface is shown in figure 4. The radius vector has the form (5). In such a case we have the following dependencies between the Cartesian  $x_i$  and the parabolic coordinates  $\alpha_i$  for  $i = 1, 2, 3$  on the parabolic surface [4]:

$$x_1 = a \cos^2(\alpha_3 \Lambda) \cos \alpha_1, \quad x_2 = a \cos^2(\alpha_3 \Lambda) \sin \alpha_1, \quad x_3 = \frac{1}{\Lambda} \sin(\alpha_3 \Lambda), \quad (12)$$

whereas

$$0 \leq \alpha_1 < \alpha_e < 2\pi, \quad |\alpha_3| \leq \frac{1}{\Lambda} \arccos \sqrt{\frac{a-w}{a}}, \quad \Lambda \equiv \frac{1}{b} \sqrt{\frac{w}{a}}.$$

We denote:  $a$  – the largest radius,  $a_1$  – the smallest radius,  $2b$  – the bearing length,  $w = a - a_1$ . Equations (12) satisfy the following equation of parabolic surface:

$$x_1^2 + x_2^2 = \left[ a - \left( \frac{x_3}{b} \right)^2 w \right]^2. \quad (13)$$

Neglecting the terms of the order  $\psi = \varepsilon/a \approx 10^{-3}$ , we obtain the parabolic Lamé coefficients of bone surface in the following final form [3]:

$$h_1 = a \cos^2(\alpha_3 \Lambda), \quad h_2 = 1, \quad h_3 = \sqrt{1 + 4a^2 \Lambda^2 \sin^2(\alpha_3 \Lambda) \cos^2(\alpha_3 \Lambda)}. \quad (14)$$

#### 4. Mathematical description of the synovial fluid viscosity

Viscosity of synovial fluid with non-Newtonian properties has been examined experimentally by Dowson [5]. Using the experimental values [5] we obtain the approximation formulae for dynamic viscosity values, i.e. [4]:

$$\eta_p \equiv \eta_\infty + \frac{\Delta\eta}{1 + A\theta + B\theta^2}, \quad \theta \equiv \frac{\partial v_1}{\partial \alpha_2}, \quad (15)$$

where  $\Delta\eta \equiv \eta_0 - \eta_\infty$ ,  $\eta_\infty$  – dynamic viscosity of synovial fluid for high shear rate, i.e. for  $\theta \approx 1000 \text{ s}^{-1}$ ,  $\eta_0$  – dynamic viscosity of synovial fluid for low shear rate, i.e. for  $\theta \approx 5 \text{ s}^{-1}$ .

Moreover, the values  $A$ ,  $B$  denote experimental coefficients obtained by means of Cooke's and Dowson's experiments [5]. These values are obtained both for normal and pathological joints.

**THEOREM.** *If dynamic viscosity function of synovial non-Newtonian fluid has the form of approximation (15) obtained from Dowson's experiments and if we consider the axisymmetrical flow in a narrow gap between two rotational surfaces, then dynamic viscosity is constant only in the gap height direction  $\alpha_2$ .*

**PROOF.** We insert formula (15) into simplified equation of conservation of momentum (1). Differentiating this expression with respect to  $\alpha_2$  we obtain:

$$\frac{\partial^2 v_1}{\partial \alpha_2^2} \left\{ \eta_\infty + \Delta\eta \frac{1 - B \left( \frac{\partial v_1}{\partial \alpha_2} \right)^2}{\left[ 1 + A \frac{\partial v_1}{\partial \alpha_2} + B \left( \frac{\partial v_1}{\partial \alpha_2} \right)^2 \right]^2} \right\} = 0. \quad (16)$$

For the normal synovial fluid  $A = 1.88307 \text{ s}$ ,  $B = 0.00458 \text{ s}^2$ , whereas for pathological joint we have  $A = 0.03345 \text{ s}$ ,  $B = 0.00131 \text{ s}^2$ . Thus the expression in braces in equation (16) is not equal to zero for  $0 < \theta \leq 1000 \text{ s}^{-1}$  and for  $0.400 \leq \eta_0 \leq 100 \text{ Pa}\cdot\text{s}$ ,  $0.01 \leq \eta_\infty \leq 0.400 \text{ Pa}\cdot\text{s}$ . Without loss of generality we have from equation (16):

$$\frac{\partial^2 v_1}{\partial \alpha_2^2} = 0. \quad (17)$$

Integrating twice equation (17) with respect to the variable  $\alpha_2$  we obtain:

$$v_1 = C_1 \alpha_2 + C_2. \quad (18)$$

$C_1$  and  $C_2$  are independent constants of the variable  $\alpha_2$ .

Taking into account solution (18), then the shear rate defined in equation (15) gives:

$$\theta = \frac{\partial v_1}{\partial \alpha_2} = C_1. \quad (19)$$

Thus the shear rate  $\theta$  is independent of  $\alpha_2$ . Hence, by virtue of equation (15), the dynamic viscosity of synovial fluid is independent of  $\alpha_2$ , which completes the proof of the Theorem.

## 5. A sketch of solution method

The synovial flow is generated by rotation of a bone. Cartilage is motionless, thus:

$$\begin{aligned} v_1 = \omega h_1, \quad v_2 = 0, \quad v_3 = 0 \quad \text{for} \quad \alpha_2 = 0 \quad (\text{bone surface}), \\ v_1 = v_2 = v_3 = 0 \quad \text{for} \quad \alpha_2 = \varepsilon(\alpha_3) \quad (\text{cartilage surface}). \end{aligned} \quad (20)$$

If we take into account the above boundary conditions for  $v_1$  and  $v_2$ , from equations (1) and (3) we obtain the particular solutions of velocity components:  $v_1(p)$  and  $v_3(p)$ , which depend on the pressure  $p$ . Solutions  $v_1(p)$  and  $v_3(p)$  are next inserted into continuity equation (4). Continuity equation (4) is integrated with respect to the variable  $\alpha_2$ . Imposing the condition  $v_2 = 0$  for  $\alpha_2 = 0$  upon the synovial fluid velocity component in the direction of gap height, we obtain general solution  $v_2(p)$ . The boundary condition  $v_2 = 0$  for  $\alpha_2 = \varepsilon(\alpha_3)$  imposed upon the general solution  $v_2(p)$  leads to the modified Reynolds equation in a reduced form. From this reduced, ordinary differential Reynolds equation we find a pressure function in its final form  $p(\alpha_3)$ . The pressure function  $p(\alpha_3)$  is inserted into previously obtained synovial fluid velocity components:  $v_1(p)$ ,  $v_2(p)$  and  $v_3(p)$ . Hence, we arrive at final analytical solutions.

## 6. Analytical solutions for the axisymmetrical flow between two rotational half-spherical bone surfaces and variable gap height

Bone head in a human hip joint is simplified to the hemispherical shape [3], [4] (see figure 5). In this case, the flow of synovial fluid in a narrow gap is considered to be axisymmetrical, thus the spherical coordinate system will be given in the following form [4]:

$$\alpha_1 \equiv \phi, \quad \alpha_2 = \varepsilon_0 \alpha_{21}, \quad \alpha_3 = R_0 \chi_1. \quad (21)$$

The Lamé coefficient is given by formulae (7), where  $\chi_1 = \alpha_3/R_0$  and  $\chi_1 \in (0, \pi/2)$  is the dimensionless longitudinal coordinate. Variable gap height has the following form:  $\varepsilon(\chi_1) = \varepsilon_0 \varepsilon_1(\chi_1)$ . We define  $y_1 \equiv \alpha_{21}/\varepsilon_1(\chi_1)$ , where  $\alpha_{21} \leq \varepsilon_1(\chi_1)$ .  $R_0$  denotes the radius of the sphere.

The motion of the head of bone implies the synovial flow in the gap. Hence, on the surface of spherical head of the bone we assume for  $\alpha_2 = 0$ :

$$v_1 = \omega h_1 = \omega R_0 \sin \chi_1, \quad v_2 = 0, \quad v_3 = 0. \quad (22)$$

Symbol  $\omega$  denotes an angular velocity of the spherical head.

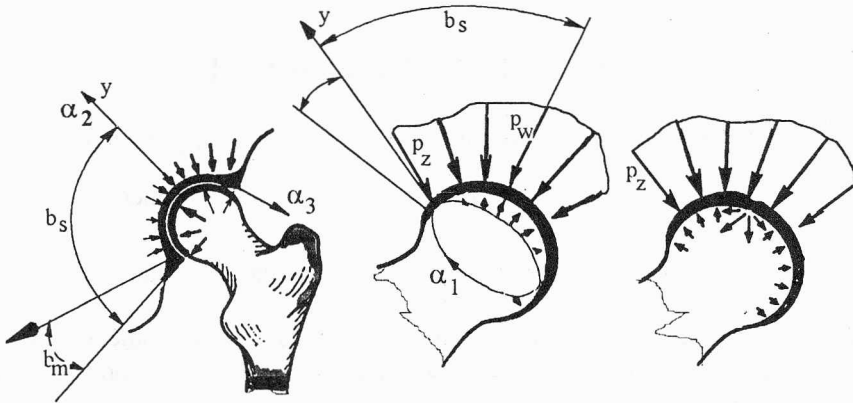


Fig. 5. Human hip joint

The cartilage is motionless, therefore for  $\alpha_2 = \varepsilon$  we obtain

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0. \quad (23)$$

Symbol  $\varepsilon$  denotes a constant gap height.

The pressure distribution counteracts the load of a joint (see figure 5). We arrive at the following boundary conditions in the inlet and outlet of the gap, respectively:

$$p = p_z, \quad \alpha_3 = b_m, \quad p = p_w, \quad \alpha_3 = b_s, \quad (24)$$

where  $p_z$  denotes the pressure at the inlet into the articulation gap, and  $p_w$  – the pressure inside the human joint gap (see figure 5). The system of equations (1)–(4) for assumptions (21) and boundary conditions (22), (24) has following analytical solutions:

$$v_1 = \omega R_0 (1 - y_1) \sin \chi_1, \quad (25)$$

$$v_2(\alpha_{21}, \chi_1) = -\frac{1}{2}(p_z - p_w) \frac{\varepsilon_0^3}{\eta_0 R_0^2} y_1^2 (y_1 - 1) \frac{\frac{1}{\varepsilon_1(\chi_1)} \frac{\partial \varepsilon_1(\chi_1)}{\partial \chi_1} \frac{1}{\sin \chi_1}}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}(\chi_1)}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1}$$



$$\begin{aligned}
 & + \frac{1}{60} \frac{\rho \omega^2 \varepsilon_0^3}{\eta_0} \frac{y_1^2 (y_1 - 1)}{\sin \chi_1} \left\{ \frac{d}{d\chi_1} \left( \frac{\varepsilon_1^3(\chi_1) \sin^2 \chi_1 \cos \chi_1}{\eta_{p1}(\chi_1)} \right) (y_1 - 1)(y_1 - 3) \right. \\
 & \quad \left. - \left[ \frac{9 \sin^2(b_{s1}) - \sin^2(b_{m1})}{2} \frac{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1}{\left( \frac{\varepsilon_1^3(\chi_1)}{\eta_{p1}(\chi_1)} \sin^2 \chi_1 \cos \chi_1 \right)} \right] \right. \\
 & \quad \left. \times (5y_1^2 - 15y_1 + 6) \frac{1}{\varepsilon_1(\chi_1)} \frac{\partial \varepsilon_1}{\partial \chi_1} \right\}, \tag{26}
 \end{aligned}$$

$$\begin{aligned}
 v_3(\alpha_{21}, \chi_1) = & -\frac{1}{2} (p_z - p_w) \frac{\varepsilon_0^2}{\eta_0 R_0} \frac{\frac{1}{\varepsilon_1(\chi_1)} \frac{1}{\sin \chi_1}}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1} y_1 (y_1 - 1) \\
 & + \frac{\rho \omega^2 \varepsilon_0^2 R_0}{20 \eta_0} \left\{ -\frac{3}{2} (\sin^2 b_{s1} - \sin^2 b_{m1}) \frac{\frac{1}{\varepsilon_1(\chi_1)} \frac{1}{\sin \chi_1}}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}(\chi_1)}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1} \right. \\
 & \quad \left. - \frac{1}{3} \frac{\varepsilon_1^2 \sin \chi_1 \cos \chi_1}{\eta_{p1}(\chi_1)} (5y_1^2 - 15y_1 + 6) \right\} y_1 (y_1 - 1), \tag{27}
 \end{aligned}$$

$$\begin{aligned}
 p(\chi_1) = & p_w + (p_z - p_w) \frac{\int_{\chi_1}^{b_{s1}} \frac{\eta_{p1}(\chi_1)}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}(\chi_1)}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1} + \frac{3}{20} \rho \omega^2 R^2 \left[ (\sin^2 b_{s1} - \sin^2 b_{m1}) \right. \\
 & \quad \left. \times \frac{\int_{\chi_1}^{b_{s1}} \frac{\eta_{p1}(\chi_1)}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1}(\chi_1)}{\varepsilon_1^3(\chi_1) \sin \chi_1} d\chi_1} - (\sin^2 b_{s1} - \sin^2 \chi_1) \right], \tag{28}
 \end{aligned}$$

where:

$$\eta_{p1} = \frac{\eta_\infty}{\eta_0} + \frac{\frac{\Delta\eta}{\eta_0}}{1 + A\theta + B\theta^2}, \quad \theta \cong \frac{\omega R_0}{\varepsilon} \sin \chi_1, \quad (29)$$

whereas

$$0 < b_{m1} \equiv \frac{b_m}{R_0} \leq \chi_1 \leq b_{s1} \equiv \frac{b_s}{R_0} < \frac{\pi}{2},$$

$$0 \leq y_1 \leq 1, \quad 0 \leq \alpha_1 < \alpha_e < 2\pi, \quad p_w = O\left(\frac{\eta_\infty^2}{\rho\varepsilon^2}\right).$$

We show the above terms of the fluid velocity components, resulting both from the motion of the head of the bone (see terms multiplied by the factor  $\omega$ ) and from the pressure difference (see terms multiplied by the difference  $p_z - p_w$ ).

### 7. Analytical solutions for the axisymmetrical flow between two rotational hyperbolic bone surfaces and the gap of variable height

In this case, the flow of synovial fluid in a narrow gap is axisymmetrical, thus the hyperbolic coordinate system will take the form (see figure 6):

$$\alpha_1 \equiv \alpha_{11}, \quad \alpha_2 = \varepsilon_0 \alpha_{21}, \quad \alpha_3 = \frac{1}{A} \alpha_{31}. \quad (30)$$

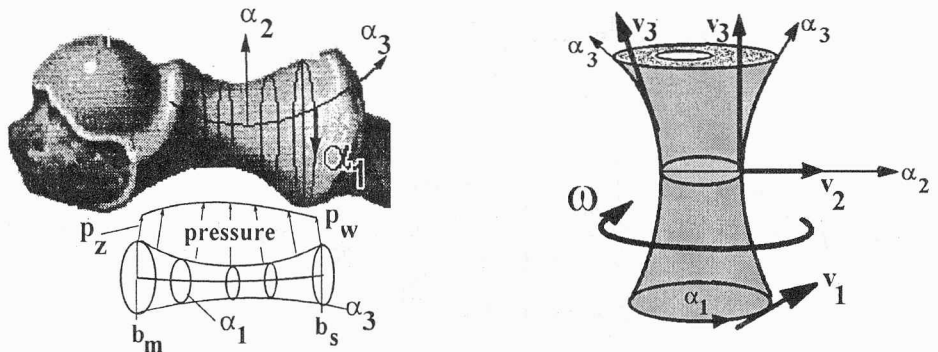


Fig. 6. Radial elbow joint in hyperbolic coordinates

The values of the Lamé coefficients are given by equation (11) and  $0 \leq \alpha_1 \leq \alpha_e < 2\pi$ ,  $0 \leq \alpha_2 \leq \varepsilon(\alpha_3)$ ,  $b_m \leq \alpha_3 \leq b_s$ . Dimensionless gap height and modified gap height coordinate have the form:

$$\varepsilon_1(\alpha_{31}) = \frac{\varepsilon(\alpha_3)}{\varepsilon_0}, \quad y_1 \equiv \frac{\alpha_{21}}{\varepsilon_1(\alpha_{31})}. \quad (31)$$

The motion of the head of bone implies the synovial flow in the gap. Hence, on the surface of hyperbolic head of the bone we assume for  $\alpha_2 = 0$ :

$$v_1 = \omega h_1 = \omega a \sec^2(\alpha_{31}), \quad v_2 = 0, \quad v_3 = 0, \quad (32)$$

where  $\omega$  is the angular velocity of the hyperbolic head.

The cartilage is motionless, therefore for  $\alpha_2 = \varepsilon$ :

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0. \quad (33)$$

Symbol  $\varepsilon(\alpha_{31})$  denotes the gap height. Pressure boundary conditions have the same form as formula (24).

In hyperbolic curvilinear coordinates, the velocity components of synovial fluid have the following form [1], [4], [5], [6]:

$$v_1 = \omega a (1 - y_1) \sec^2(\alpha_{31}), \quad (34)$$

$$\begin{aligned} v_2(\alpha_{21}, \alpha_{31}) = & -\frac{1}{2}(p_z - p_w) \frac{\varepsilon_0^3}{\eta_0} \frac{a_1 - a}{b^2 a} y_1^2 (y_1 - 1) \frac{\partial \varepsilon_1(\alpha_{31})}{\partial \alpha_{31}} \frac{\cos^4(\alpha_{31})}{\varepsilon_1(\alpha_{31}) \Omega_h \int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}} \\ & + \frac{1}{60} \rho \omega^2 a \frac{\varepsilon_0^3}{\eta_0} \frac{a_1 - a}{b^2} y_1^2 (y_1 - 1) \left\{ 2(y_1 - 1)(y_1 - 3) \frac{\partial}{\partial \alpha_{31}} \left[ \frac{\varepsilon_1^3(\alpha_{31}) \sin(\alpha_{31})}{\eta_{p1} \Omega_h \cos^5(\alpha_{31})} \right] \right. \\ & \left. - \left[ \frac{9 \sec^4(b_{s1}) - \sec^4(b_{m1})}{2 \int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}} \right] \right. \\ & \left. + 2(5y_1^2 - 15y_1 + 6) \frac{\varepsilon_1^3(\alpha_{31}) \sin(\alpha_{31})}{\eta_{p1} \Omega_h \cos^5(\alpha_{31})} \right] \frac{1}{\varepsilon_1(\alpha_{31})} \frac{\partial \varepsilon_1}{\partial \alpha_{31}} \left\} \frac{\cos^4(\alpha_{31})}{\Omega_h}, \quad (35) \end{aligned}$$

$$v_3(\alpha_{21}, \alpha_{31}) = -\frac{1}{2}(p_z - p_w) \frac{\varepsilon_0^2}{\eta_0 b} \sqrt{\frac{a_1 - a}{a}} y_1 (y_1 - 1) \frac{\cos^2(\alpha_{31})}{\varepsilon_1(\alpha_{31}) \int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}}$$

$$-\frac{3}{20} \rho \omega^2 a^2 \frac{\varepsilon_0^2}{\eta_0 b} \sqrt{\frac{a_1 - a}{a}} y_1 (y_1 - 1) \frac{\cos^2(\alpha_{31})}{\varepsilon_1(\alpha_{31})} \left\{ \frac{1}{2} \frac{\sec^4(b_{s1}) - \sec^4(b_{m1})}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}} \right. \\ \left. + \frac{2}{9} (5y_1^2 - 15y_1 + 6) \frac{\varepsilon_1^3(\alpha_{31})}{\eta_{p1}(\alpha_{31})} \frac{1}{\Omega_h} \frac{\sin(\alpha_{31})}{\cos^5(\alpha_{31})} \right\}, \quad (36)$$

$$p(\alpha_{31}) = p_w + (p_z - p_w) \frac{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}} + \frac{3}{20} \rho \omega^2 a^2 \left\{ [\sec^4(b_{s1}) - \sec^4(b_{m1})] \right. \\ \left. \times \frac{\int_{b_{s1}}^{\alpha_{31}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_h}{\varepsilon_1^3(\alpha_{31})} d\alpha_{31}} + [\sec^4(b_{s1}) - \sec^4(\alpha_{31})] \right\}, \quad (37)$$

where:

$$\frac{b_m}{b} \sqrt{\frac{a_1 - a}{a}} = b_{m1} \equiv b_m \Lambda \leq \alpha_{31} \leq b_s \Lambda \equiv b_{s1} = \frac{b_s}{b} \sqrt{\frac{a_1 - a}{a}}, \\ 0 \leq \alpha_{11} \leq \alpha_e < 2\pi, \quad 0 \leq \alpha_{21} \equiv \frac{\alpha_2}{\varepsilon_0} \leq \varepsilon_1(\alpha_{31}).$$

The symbols  $b_{m1}$  and  $b_{s1}$  denote the upper and lower dimensionless limits of lubrication, respectively,  $a$  is the smallest radius of the cross-section of hyperboloid,  $a_1$  denotes the largest radius of cross-section of hyperboloid, and  $2b$  is the bone length. Moreover, we introduce the following notations:

$$0 \leq y_1 \equiv \frac{\alpha_{21}}{\varepsilon_1(\alpha_{31})} \leq 1 \quad \text{and} \quad \Omega_h \equiv \sqrt{1 + 4 \frac{a(a_1 - a)}{b^2} \tan^2(\alpha_{31})}. \quad (38)$$

Dynamic viscosity of synovial fluid has the following form:

$$\eta_p = \eta_0 \eta_{p1}(\alpha_{31}) = \eta_0 \left( \frac{\eta_\infty}{\eta_0} + \frac{\frac{\Delta \eta}{\eta_0}}{1 + A\theta + B\theta^2} \right). \quad (39)$$

The shear rate is described by the following expression:

$$\theta \equiv \frac{\partial v_1}{\partial \alpha_2} = -\frac{\omega a}{b} \sqrt{\frac{a_1 - a}{a}} \frac{\sec^2(\alpha_{31})}{\alpha_{31}}. \quad (40)$$

### 8. Analytical solutions for the axisymmetrical flow between two rotational parabolic bone surfaces and the gap of variable height

In this case, the flow of synovial fluid in a narrow gap is considered as axisymmetrical, thus the parabolic coordinate system will be taken in the form (see figures 7 and 8):

$$\alpha_1 \equiv \alpha_{11}, \quad \alpha_2 = \varepsilon_0 \alpha_{21}, \quad \alpha_3 = \frac{1}{\Lambda} \alpha_{31}. \quad (41)$$

The values of the Lamé coefficients are given by expressions (14), and  $0 \leq \alpha_1 \leq \alpha_e \leq 2\pi$ ,  $0 \leq \alpha_2 \leq \varepsilon(\alpha_3)$ ,  $b_m \leq \alpha_3 \leq b_s$ . Dimensionless gap height has the same form as in expression (31).

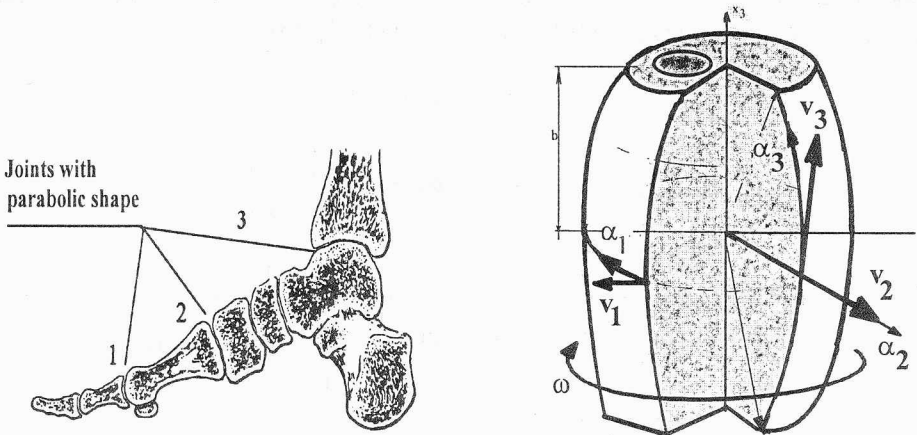


Fig. 7. Joint in parabolic coordinates. Fibular section by the foot across the finger I

The motion of the parabolic head of a bone implies the synovial flow in the gap. Hence, on the surface of parabolic head of the bone we assume for  $\alpha_2 = 0$ :

$$v_1 = \omega h_1 = \omega a \cos^2(\alpha_{31}), \quad v_2 = 0, \quad v_3 = 0, \quad (42)$$

where  $\omega$  denotes the angular velocity of the parabolic head.

The cartilage is motionless, therefore for  $\alpha_2 = \varepsilon$ :  $v_1 = 0$ ,  $v_2 = 0$ ,  $v_3 = 0$ .

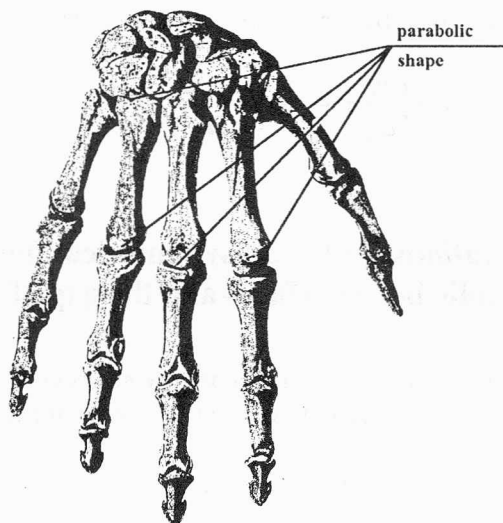


Fig. 8. Joint in human hand

Symbol  $\varepsilon(\alpha_{31})$  denotes the gap height. Pressure boundary conditions have the same form as formula (24). In parabolic curvilinear coordinates the synovial fluid velocity components have the following form [1], [4], [5], [6]:

$$v_1 = \omega a(1 - y_1) \cos^2(\alpha_{31}), \quad (43)$$

$$v_2(\alpha_{21}, \alpha_{31}) = -\frac{1}{2}(p_z - p_w) \frac{\varepsilon_0^3}{\eta_0} \frac{a - a_1}{b^2 a} y_1^2$$

$$\times (y_1 - 1) \frac{\partial \varepsilon_1(\alpha_{31})}{\partial \alpha_{31}} \frac{\sec^3(\alpha_{31})}{\varepsilon_1(\alpha_{31}) \Omega_p \int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3} \sec(\alpha_{31}) d\alpha_{31}}$$

$$-\frac{1}{60} \rho \omega^2 a \frac{\varepsilon_0^3}{\eta_0} \frac{a - a_1}{b^2} y_1^2 (y_1 - 1) \left\{ (y_1 - 1)(y_1 - 3) \frac{\partial}{\partial \alpha_{31}} \left[ \frac{\varepsilon_1^3(\alpha_{31})}{\eta_{p1} \Omega_p} \sin(2\alpha_{31}) \cos^3(\alpha_{31}) \right] \right.$$

$$\left. - \left[ \frac{9 \cos^4(b_{s1}) - \cos^4(b_{m1})}{2 \int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3} \sec(\alpha_{31}) d\alpha_{31}} \right] \right\}$$

$$-\left(5y_1^2 - 15y_1 + 6\right) \frac{\varepsilon_1^3(\alpha_{31}) \sin(2\alpha_{31}) \cos^3(\alpha_{31})}{\eta_{p1} \Omega_p} \left] \frac{1}{\varepsilon_1(\alpha_{31})} \frac{\partial \varepsilon_1}{\partial \alpha_{31}} \right\} \frac{\sec^3(\alpha_{31})}{\Omega_p}, \quad (44)$$

$$v_3(\alpha_{21}, \alpha_{31}) = -\frac{1}{2}(p_z - p_w) \frac{\varepsilon_0^2}{\eta_0 b} \sqrt{\frac{a-a_1}{a}} y_1$$

$$\times (y_1 - 1) \frac{\sec^2(\alpha_{31})}{\varepsilon_1(\alpha_{31}) \int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3(\alpha_{31})} \sec(\alpha_{31}) d\alpha_{31}}$$

$$-\frac{3}{20} \rho \omega^2 a^2 \frac{\varepsilon_0^2}{\eta_0 b} \sqrt{\frac{a-a_1}{a}} y_1 (y_1 - 1) \frac{\sec^2(\alpha_{31})}{\varepsilon_1(\alpha_{31})} \left\{ \frac{\cos^4(b_{s1}) - \cos^4(b_{m1})}{2 \int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3} \sec(\alpha_{31}) d\alpha_{31}} \right.$$

$$\left. -\frac{1}{9} (5y_1^2 - 15y_1 + 6) \frac{\varepsilon_1^3(\alpha_{31})}{\eta_{p1}(\alpha_{31})} \frac{1}{\Omega_p} \sin(2\alpha_{31}) \cos^3(\alpha_{31}) \right\}, \quad (45)$$

$$p(\alpha_{31}) = p_w + (p_z - p_w) \frac{\int_{\alpha_{31}}^{b_{s1}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3(\alpha_{31})} \sec^2(\alpha_{31}) d\alpha_{31}}{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3(\alpha_{31})} \sec^2(\alpha_{31}) d\alpha_{31}}$$

$$+ \frac{3}{20} \rho \omega^2 a^2 \{ [\cos^4(b_{s1}) - \cos^4(b_{m1})]$$

$$\left. \times \frac{\int_{b_{m1}}^{b_{s1}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3(\alpha_{31})} \sec^2(\alpha_{31}) d\alpha_{31}}{\int_{b_{m1}}^{\alpha_{31}} \frac{\eta_{p1} \Omega_p}{\varepsilon_1^3(\alpha_{31})} \sec^2(\alpha_{31}) d\alpha_{31}} + [\cos^4(b_{s1}) - \cos^4(\alpha_{31})] \right\}, \quad (46)$$

where:





$$\theta \equiv \frac{\partial v_1}{\partial \alpha_2} = -\frac{\omega a}{b} \sqrt{\frac{a-a_1}{a}} \frac{\cos^2(\alpha_{31})}{\alpha_{31}}. \quad (48)$$

We show the above terms of the fluid velocity components resulting both from the motion of the head of the bone, i.e. the Couette flow (see terms multiplied by the factor  $\omega$ ), and from the pressure difference, i.e. the Poiseuille flow (see terms multiplied by the difference  $p_z - p_w$ ). Further steps taken in order to evaluate the problem are presented in figure 9.

## 9. Conclusions

On the grounds of basic fluid mechanics equations and by virtue of papers [1], [2], [3], [4], [5] the final derivation of the dimensionless and dimensional synovial fluid velocity components and the final form of hydrodynamic pressure distributions in the axisymmetrical flow between two rotational bone surfaces in a narrow gap of human joint are presented. The following assumptions are accepted: variable gap height, axisymmetrical flow, variable dynamic viscosity of the synovial fluid as non-Newtonian fluid, curvilinear orthogonal coordinates, i.e. spherical, hyperbolic and parabolic.

The paper gives the method of analytical determination of the synovial, non-Newtonian fluid velocity components and the pressure distribution. The main results obtained enable the numerical solutions of the hydrodynamic lubrication problem which arises when two bone surfaces of various geometry cooperate in human joints. This paper is a continuation of [1]–[4].

The lubrication mechanism of human joints have been studied for many years and many theories have been proposed to explain the flow of synovial fluid and very low friction and wear characteristics of these joints [7]–[10]. In the present paper, final analytical solutions for symmetrical synovial flow in the form which have not been elaborated hitherto are studied.

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