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Numerical contribution to the viscoelastic magnetic lubrication of human joint in periodic motion

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This paper presents the author's numerical contribution to unsymmetrical viscoelastic hydrodynamic lubrication of human joints with synovial fluid in periodically changed time and unsteady magnetic field. We assume that bone head in human joint moves in two directions, namely in circumference and meridian directions. Basic equations describing the flow of synovial fluid in human hip joint are solved analytically and numerically. Numerical calculations are performed in Mathcad 2000 Professional Program, taking into account the method of finite differences. This method satisfies stability of numerical solutions of partial differential equations and values of capacity forces occurring in human joints.

Key words: viscoelastic lubrication, human joint, periodic motion, magnetic field

1. Preliminaries

There is a number of current studies whose authors have different approaches to the study of joint biomechanics. Lubrication of human joint under unsteady periodic conditions and for real viscoelastic properties of synovial fluid has not been examined hitherto. Viscoelastic lubrication of human joint in unsteady, periodic motion and magnetic field was not considered in the papers [1], [4]–[7], [11]–[19]. In the present study, the changes that occur during the viscoelastic lubrication of human joints under varying periodic, unsteady conditions are examined.

In the paper, we assume rotational motion of the human bone, periodic and unsteady flow of viscoelastic synovial fluid, periodic time-dependent gap height, changeable synovial fluid viscosity, variable geometry of gap height, constant density ρ_0 of synovial fluid, and isothermal, incompressible flow of synovial fluid. We also assume that bone head can make rotational motion in two directions at two various angular velocities (see figure 1). In the case of unsymmetrical flow of synovial fluid, three components v_1, v_2, v_3 of its velocity vector depend on the variables α_1, α_2, α_3, while the time t and the pressure function p depend on α_1, α_3, t. The gap height ε may be a function of the variables α_1, α_3 and the time t. The symbol α_1 denotes the co-ordinate in circumference direction, α_2 is the co-ordinate in gap height direction, α_3 stands for a generating line of rotational bone surface or co-ordinate in longitudinal direction.

2. Basic equations

The problem of lubrication of human hip joint will be presented by means of the conservation of momentum, continuity and Maxwell's equations [2], [10]:

$$\mathrm{Div}\mathbf{S} + \mu_0(\mathbf{N}\nabla)\mathbf{H} = \rho d\mathbf{v}/dt, \quad \mathrm{div}\mathbf{v} = 0, \quad \nabla^2\mathbf{H} \equiv \mu_0\varepsilon\partial^2\mathbf{H}/\partial t^2, \tag{1}$$

where: $\mathbf{S}$ – the stress tensor, $\mathbf{v}$ – synovial fluid velocity (m/s), $\mathbf{H}$ – the magnetic intensity vector (A/m) with the components (H_1, H_2, H_3), $\mathbf{N}$ – the magnetization vector (A/m) with the components (N_1, N_2, N_3), μ_0 – the magnetic permeability coefficient of free space ($\mathrm{mkgs^{-2}A^{-2}}$), ε – electric permeability coefficient of synovial fluid ($\mathrm{s^4A^2m^{-3}kg^{-1}}$). We assume that synovial fluid is a good insulator, i.e. the electric conductivity coefficient $\sigma = 0$. Moreover, the second-order approximation of the general constitutive equation given by Rivlin and Ericksen can be written in the following form [10]:

$$\mathbf{S} = -p\mathbf{I} + \eta_0\mathbf{A}_1 + \alpha(\mathbf{A}_1)^2 + \beta\mathbf{A}_2, \quad \mathbf{A}_1 \equiv \mathbf{L} + \mathbf{L}^T,$$

$$\mathbf{A}_2 \equiv \mathrm{grad}\,\mathbf{a} + (\mathrm{grad}\,\mathbf{a})^T + 2\mathbf{L}^T\mathbf{L}, \quad \mathbf{a} \equiv \mathbf{L}\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}, \tag{2}$$

where: p – pressure, $\mathbf{I}$ – the unit tensor, $\mathbf{A}_1$, and $\mathbf{A}_2$ – the first two Rivlin–Ericksen tensors, $\mathbf{L}$ – the tensor of gradient fluid velocity vector ($\mathrm{s^{-1}}$), $\mathbf{L}^T$ – the tensor of transpose of a matrix of gradient vector of a biological fluid ($\mathrm{s^{-1}}$), t – the time (s), $\mathbf{a}$ – the acceleration vector ($\mathrm{m/s^2}$). The symbols: η_0, α, β stand for three material constants of synovial fluid, where η_0 denotes dynamic viscosity (Pas), the symbol β determines the pseudoviscosity coefficient ($\mathrm{Pas^2}$) and describes the friction forces between viscoelastic particles of synovial fluid. The acceleration terms have been neglected. Only time derivatives of velocity component have been retained. The tangential and vertical acceleration of joint surface, variable in time, is taken into account. We also neglect $Re\,\Psi$ and $\Psi \equiv \varepsilon / R \approx 10^{-3}$, and the centrifugal forces, where R is the radius of curvature of bone surface. We assume that the components of magnetic intensity vector and the components of magnetisation vector are constant in the height directions of joint gap. We require curvilinear, orthogonal system of co-ordinates α_1, α_2, α_3 with the Lamé coefficients h_1, h_2, h_3, respectively. From the boundary conditions of thin layer it follows that $h_2 = 1$. After boundary simplifications the system of conservation of momentum, continuity and Maxwell's equations has the form of the system (3)–(10). Equations of motion are as follows [9], [10]:

$$\rho_0 \frac{\partial v_1}{\partial t} = -\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\left(\eta_0 \frac{\partial v_1}{\partial \alpha_2}\right) + \beta \frac{\partial^3 v_1}{\partial t \partial \alpha_2^2} + \mu_0 \frac{N_1}{h_1}\frac{\partial H_1}{\partial \alpha_1} + \mu_0 \frac{N_3}{h_3}\frac{\partial H_1}{\partial \alpha_3}, \tag{3}$$

$$0 = \frac{\partial p}{\partial \alpha_2}, \tag{4}$$

$$\rho_0 \frac{\partial v_3}{\partial t} = -\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3} + \frac{\partial}{\partial \alpha_2}\left(\eta_0 \frac{\partial v_3}{\partial \alpha_2}\right) + \beta \frac{\partial^3 v_3}{\partial t \partial \alpha_2^2} + \mu_0 \frac{N_1}{h_1}\frac{\partial H_3}{\partial \alpha_1} + \mu_0 \frac{N_3}{h_3}\frac{\partial H_3}{\partial \alpha_3}, \tag{5}$$

$$h_3 \frac{\partial v_1}{\partial \alpha_1} + h_1 h_3 \frac{\partial v_2}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3}\left(h_1 v_3\right) = 0\,. \tag{6}$$

Terms multiplied by the factor β describe the influence of viscoelastic properties of synovial fluid on the lubrication process. If the coefficient β tends to zero, then set of equations (1)–(6) tends to the equations describing the lubrication of human joints presented in papers [12]–[19]. Maxwell's equations are as follows [2]:

$$\nabla^2\mathbf{H} \equiv \text{grad}(\text{div}\mathbf{H}) - \text{rot}(\text{rot}\mathbf{H}) = \mu_{0E}\partial^2\mathbf{H}/\partial t^2, \tag{7}$$

hence:

$$X(\xi,\zeta_2,\zeta_3) \equiv \frac{1}{h_1}\frac{\partial \xi}{\partial \alpha_1} - \frac{1}{h_3}\left[\frac{\partial(h_3\zeta_3)}{\partial \alpha_2} - \frac{\partial(\zeta_2)}{\partial \alpha_3}\right] = \mu_{0E}\frac{\partial^2 H_1}{\partial t^2}, \tag{8}$$

$$Y(\xi,\zeta_1,\zeta_3) \equiv \frac{\partial \xi}{\partial \alpha_2} - \frac{1}{h_1 h_3}\left[\frac{\partial(h_1\zeta_1)}{\partial \alpha_3} - \frac{\partial(h_3\zeta_3)}{\partial \alpha_1}\right] = \mu_{0E}\frac{\partial^2 H_2}{\partial t^2}, \tag{9}$$

$$Z(\xi,\zeta_1,\zeta_2) \equiv \frac{1}{h_3}\frac{\partial \xi}{\partial \alpha_3} - \frac{1}{h_1}\left[\frac{\partial(\zeta_2)}{\partial \alpha_1} - \frac{\partial(\zeta_1 h_1)}{\partial \alpha_2}\right] = \mu_{0E}\frac{\partial^2 H_3}{\partial t^2}, \tag{10}$$

where:

$$\zeta_1 \equiv \frac{1}{h_3}\left[\frac{\partial(H_3 h_3)}{\partial \alpha_2} - \frac{\partial(H_2)}{\partial \alpha_3}\right], \quad \zeta_2 \equiv \frac{1}{h_1 h_3}\left[\frac{\partial(H_1 h_1)}{\partial \alpha_3} - \frac{\partial(H_3 h_3)}{\partial \alpha_1}\right],$$

$$\zeta_3 \equiv \frac{1}{h_1}\left[\frac{\partial(H_2)}{\partial \alpha_1} - \frac{\partial(H_1 h_1)}{\partial \alpha_2}\right]. \tag{11}$$

We denote: $\mu_{0\varepsilon} \equiv \mu_0 \in 0 < \alpha_1 \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \le \varepsilon$, H_i are the components of magnetic intensity vector **H** (A/m), $\xi \equiv \text{div}\mathbf{H}$, $B_i = \mu_0(H_i + N_i)$ are the components of magnetic induction vector **B** in *T*, $N_i = \chi H_i$ are the components of magnetisation vector **N** (A/m), χ is dimensionless magnetic susceptibility of synovial fluid. In order to derive the solutions of the above set of equations, an oscillating periodic motions will be discussed.

3. The method of integration

For the velocity components and the pressure, without loss the generality, the following approach has been introduced [8], [9]:

$$v_i = v_i^{(0)}(\alpha_1, \alpha_2, \alpha_3) + \sum_{k=1}^{\infty} v_i^{(k)}(\alpha_1, \alpha_2, \alpha_3)\exp(\mathbf{i}k\omega_0 t), \quad i = 1, 2, 3, \tag{12}$$

$$p = p^{(0)}(\alpha_1, \alpha_3) + \sum_{k=1}^{\infty} p^{(k)}(\alpha_1, \alpha_3)\exp(\mathbf{i}k\omega_0 t), \tag{13}$$

$$H_i = H_i^{(0)}(\alpha_1, \alpha_3) + \sum_{k=1}^{\infty} \frac{1}{2^k} H_i^{(k)}(\alpha_1, \alpha_3)\exp(\mathbf{i}k\omega_0 t), \quad i = 1, 3, \tag{14}$$

where: ω_0 is an angular velocity (s^{-1}) describing periodic perturbations in unsteady flow of synovial fluid and magnetic field in joint gap and $\mathbf{i} \equiv \sqrt{-1}$ is an imaginary unit.

Gap height has the following form:

$$\varepsilon_{\text{tot}} \equiv \varepsilon^{(0)} + \tilde{\varepsilon} \equiv \varepsilon^{(0)}(\alpha_1, \alpha_3) + \sum_{k=1}^{\infty} \varepsilon^{(k)}(\alpha_1, \alpha_3)\exp(\mathbf{i}k\omega_0 t), \tag{15}$$

where: $\tilde{\varepsilon}$ denotes time-dependent perturbation of the gap height caused by unsteady work conditions, $\varepsilon^{(k)}$ – time-independent coefficient of perturbations of gap height, $\varepsilon^{(0)}$ – time-independent primary gap height, and ε_{tot} – the total value of the gap height. Because of linear form of equations (3)–(7) a separation of a steady flow from an unsteady flow of synovial fluid is possible. We insert series (12)–(14) into the set of equations (3)–(7) and we equate terms of the same upper indexes in brackets and the same powers of exp functions. Equations of motion for steady conditions in steady magnetic field and Newtonian fluid have the form [9]:

$$0 = -\frac{1}{h_1}\frac{\partial p^{(0)}}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\left(\eta_0 \frac{\partial v_1^{(0)}}{\partial \alpha_2}\right) + M_1^{(0)}(H^{(0)}), \tag{16}$$

$$0 = \frac{\partial p^{(0)}}{\partial \alpha_2}, \tag{17}$$

$$0 = -\frac{1}{h_3}\frac{\partial p^{(0)}}{\partial \alpha_3} + \frac{\partial}{\partial \alpha_2}\left(\eta_0 \frac{\partial v_3^{(0)}}{\partial \alpha_2}\right) + M_3^{(0)}(H^{(0)}), \tag{18}$$

$$h_3 \frac{\partial v_1^{(0)}}{\partial \alpha_1} + h_1 h_3 \frac{\partial v_2^{(0)}}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3}\left(h_1 v_3^{(0)}\right) = 0, \tag{19}$$

for $0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \equiv r \le \varepsilon$.

The system of equations (16)–(19) determines an unknown pressure function $p^{(0)}$ and the unknown components $v_1^{(0)}, v_2^{(0)}, v_3^{(0)}$ of oil velocity vector in the directions α_1, α_2, α_3, respectively.

Maxwell's equations for steady conditions have the form [2]:

$$X\left(\xi^{(0)}, \zeta_2^{(0)}, \zeta_3^{(0)}\right) = 0, \quad Y\left(\xi^{(0)}, \zeta_1^{(0)}, \zeta_3^{(0)}\right) = 0, \quad Z\left(\xi^{(0)}, \zeta_1^{(0)}, \zeta_2^{(0)}\right) = 0, \tag{20}$$

at $0 < \alpha_1 \leq 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \leq \alpha_3 \leq \pi R/2 \equiv b_s$, $0 \leq \alpha_2 \leq \varepsilon$. The system of equations (20) determines the unknown components $H_1^{(0)}, H_2^{(0)}, H_3^{(0)}$ of magnetic intensity vector in the directions α_1, α_2, α_3, respectively. Equations of motion of the k steps of correction values for the unsteady periodic motion and conditions have the form [8], [9]:

$$\mathbf{i}k\omega_0\rho_0 v_1^{(k)} = -\frac{1}{h_1}\frac{\partial p^{(k)}}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\left(\eta_k \frac{\partial v_1^{(k)}}{\partial \alpha_2}\right) + \frac{1}{2^k} M_1^{(k)}(H^{(k)}), \tag{21}$$

$$0 = \frac{\partial p^{(k)}}{\partial \alpha_2}, \tag{22}$$

$$\mathbf{i}k\omega_0\rho_0 v_3^{(k)} = -\frac{1}{h_3}\frac{\partial p^{(k)}}{\partial \alpha_3} + \frac{\partial}{\partial \alpha_2}\left(\eta_k \frac{\partial v_3^{(k)}}{\partial \alpha_2}\right) + \frac{1}{2^k} M_3^{(k)}(H^{(k)}), \tag{23}$$

$$h_3 \frac{\partial v_1^{(k)}}{\partial \alpha_1} + h_1 h_3 \frac{\partial v_2^{(k)}}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3}\left(h_1 v_3^{(k)}\right) = 0 \tag{24}$$

for $k = 1, 2, 3, \ldots$ $0 < \alpha_1 \leq 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \leq \alpha_3 \leq \pi R/2 \equiv b_s$, $0 \leq \alpha_2 \leq \varepsilon$.

The symbol:

$$\eta_k \equiv \eta_0 + \mathbf{i}k\omega_0\beta \tag{25}$$

denotes an apparent viscosity (Pas). This viscosity depends on the velocity deformations caused by the viscoelastic properties (see the coefficient β) and by the angular velocity ω_0 describing periodic perturbations. This fact indicates that synovial fluid has non-Newtonian properties. If the coefficient β tends to zero, then viscoelastic and non-Newtonian fluid properties are neglected.

The system of equations (21)–(24) determines the unknown corrections $p^{(k)}$ of pressure function and the unknown corrections $v_1^{(k)}, v_2^{(k)}, v_3^{(k)}$ of the components of oil velocity vector in the directions α_1, α_2, α_3, respectively. The Maxwell equations for the k steps of corrections caused by unsteady conditions have the form [2]:

$$X\left(\xi^{(k)}, \zeta_2^{(k)}, \zeta_3^{(k)}\right) \equiv k^2\omega_0^2\mu_{0E} H_1^{(k)}, \quad Y\left(\xi^{(k)}, \zeta_1^{(k)}, \zeta_3^{(k)}\right) \equiv k^2\omega_0^2\mu_{0E} H_2^{(k)},$$

$$Z\left(\xi^{(k)}, \zeta_1^{(k)}, \zeta_2^{(k)}\right) \equiv k^2\omega_0^2\mu_{0E} H_3^{(k)} \tag{26}$$

for $k = 1, 2, 3, \ldots$ $0 \le \alpha_1 \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \le \varepsilon$.

The system of equations (26) determines the unknown corrections $H_1^{(k)}, H_2^{(k)}, H_3^{(k)}$ of the components of magnetic intensity vector in the directions α_1, α_2, α_3, respectively. Before calculations we must insert the following expressions [2] into equations (20) and (26):

$$\xi^{(k)} \equiv (\operatorname{div} \mathbf{H})^{(k)} = \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial \alpha_1}\left(h_3 H_1^{(k)}\right) + \frac{\partial}{\partial \alpha_2}\left(h_1 h_3 H_2^{(k)}\right) + \frac{\partial}{\partial \alpha_3}\left(h_1 H_3^{(k)}\right)\right], \tag{27}$$

$$\zeta_1^{(k)} \equiv \frac{1}{h_3}\left[\frac{\partial\left(H_3^{(k)} h_3\right)}{\partial \alpha_2} - \frac{\partial\left(H_2^{(k)}\right)}{\partial \alpha_3}\right], \qquad \zeta_2^{(k)} \equiv \frac{1}{h_1 h_3}\left[\frac{\partial\left(H_1^{(k)} h_1\right)}{\partial \alpha_3} - \frac{\partial\left(H_3^{(k)} h_3\right)}{\partial \alpha_1}\right],$$

$$\zeta_3^{(k)} \equiv \frac{1}{h_1}\left[\frac{\partial\left(H_2^{(k)}\right)}{\partial \alpha_1} - \frac{\partial\left(H_1^{(k)} h_1\right)}{\partial \alpha_2}\right]$$

for $k = 0, 1, 2, 3, \ldots$ $0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \equiv r \le \varepsilon$.

The functions $M_1^{(0)}, M_3^{(0)}$ in equations (16)–(18) and functions $M_1^{(k)}, M_3^{(k)}$ for $k = 1, 2, 3, \ldots$ in equations (21)–(23) have the following forms:

$$M_i^{(k)}(H^{(k)}) \equiv \mu_0 \chi \sum_{n=0}^{k}\left(\frac{H_1^{(k-n)}}{h_1}\frac{\partial H_i^{(n)}}{\partial \alpha_1} + \frac{H_3^{(k-n)}}{h_3}\frac{\partial H_i^{(n)}}{\partial \alpha_3}\right) \tag{28}$$

for $i = 1, 3$, $k = 0, 1, 2, 3, \ldots$ $0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \equiv r \le \varepsilon$.

The functions $H_1^{(0)}, H_2^{(0)}, H_3^{(0)}$ obtained from equations (20) and the functions $H_1^{(k)}, H_2^{(k)}, H_3^{(k)}$ for $k = 1, 2, 3, \ldots$ obtained from equations (26) are inserted into expression (28). Such functions are inserted into equations (16), (18), (21), (23). Afterwards the system of equations (16)–(19) and system of equations (21)–(24) are solved in order to determine the unknown functions $p^{(0)}$, $v_1^{(0)}$, $v_2^{(0)}$, $v_3^{(0)}$, $p^{(k)}$, $v_1^{(k)}$, $v_2^{(k)}$, $v_3^{(k)}$ for $k = 1, 2, 3, \ldots$

4. Boundary conditions

Bone head develops the angular velocities ω_1 and ω_3 in the directions α_1 and α_3, respectively. Acetabulum moves in circumferential α_1 and meridional α_3 directions. The gap height changes in the time in vertical direction. Moreover, it is assumed that tangential acceleration of bone head surface varies in time. Hence, for the system of equations (16)–(19) and (21)–(24) at i = 1, 3; k = 1, 2, 3, ..., the boundary conditions[8], [9], [20] are as follows:

$$v_1^{(0)}(\alpha_1, \alpha_2 = 0, \alpha_3) = U_{10}(\alpha_3), \quad v_3^{(0)}(\alpha_1, \alpha_2 = 0, \alpha_3) = U_{30}(\alpha_1), \tag{29}$$

$$v_2^{(0)}(\alpha_1, \alpha_2 = 0, \alpha_3) = 0, \tag{30}$$

$$v_i^{(0)}(\alpha_1, \alpha_2 = \varepsilon, \alpha_3) = 0 \quad \text{for} \quad i = 1, 2, 3, \tag{31}$$

$$v_1^{(k)}(\alpha_1, \alpha_2 = 0, \alpha_3) = U_{1k}(\alpha_3), \quad v_3^{(k)}(\alpha_1, \alpha_2 = 0, \alpha_3) = U_{3k}(\alpha_1), \tag{32}$$

$$v_2^{(k)}(\alpha_1, \alpha_2 = 0, \alpha_3) = 0, \tag{33}$$

$$v_i^{(k)}(\alpha_1, \alpha_2 = \varepsilon, \alpha_3) = V_{ik} \quad \text{for} \quad i = 1, 3. \tag{34}$$

We assume:

$$\sum_{k=1}^{\infty} v_2^{(k)}(\alpha_1, \alpha_2 = \varepsilon, \alpha_3) \exp(\mathbf{i}k\omega_0 t) = \frac{\partial \varepsilon_{\text{tot}}}{\partial t} = \sum_{k=1}^{\infty} \varepsilon^{(k)} \mathbf{i}k\omega_0 \exp(\mathbf{i}k\omega_0 t), \tag{35}$$

hence

$$v_2^{(k)}(\alpha_1, \alpha_2 = \varepsilon, \alpha_3) = \varepsilon^{(k)} \mathbf{i}k\omega_0 \quad \text{for} \quad k = 1, 2, 3, \ldots \tag{36}$$

Time-independent, average gap height with perturbation assumes the following form:

$$\varepsilon \equiv \Re\text{e}\,[\varepsilon^{(0)} + \frac{1}{t_m}\int_0^{t_m} (\tilde{\varepsilon})\, dt] = \varepsilon^{(0)} + \sum_{k=1}^{\infty} \left[\frac{\varepsilon^{(k)}}{t_k} \int_0^{t_m} \cos(\omega_0 k t)\, dt \right]$$
$$= \varepsilon^{(0)} + \sum_{k=1}^{\infty} \left[\varepsilon^{(k)} \frac{\sin(\omega_0 k t_m)}{\omega_0 k t_m} \right], \tag{37}$$

where t_m is an average time period of the joint gap perturbations, •e – operator of a real part of complex number. Velocities of bone and acetabulum surfaces are periodically dependent on time. Total tangential velocities of bone surface and acetabulum surface in the directions α_i have the following forms [4], [5]:

$$U_i(\alpha_i, t) = U_{i0} + \sum_{k=1}^{\infty} U_{ik} \exp(\mathbf{i}k\omega_0 t) \quad \text{(for bone)} \tag{38}$$

$$V_i(t) = \sum_{k=1}^{\infty} V_{ik} \exp(\mathbf{i}k\omega_0 t), \quad V_{ik} \equiv \frac{V_{i0}}{k^2} = \text{const} \quad \text{(for acetabulum)},$$

where U_{ik} are the time-independent coefficients of tangential velocity changes of bone surface at $i = 1, 3$, $k = 1, 2, 3, ...$, and V_{ik} are the time-independent constant coefficients of tangential velocity changes of acetabulum at $i = 1, 3$, $k = 1, 2, 3, ...$

5. Velocity of synovial fluid and pressure

5.1. Solutions for stationary flow

In the first step of solutions, we assume a stationary flow. Dynamic viscosity η_0 can be a function of α_1 and α_3 only. The system of equations (16)–(19) for boundary conditions (29) and (31) at $i = 1, 3$ has the following solutions [8], [9]:

$$v_1^{(0)} = -\frac{1}{2\eta_0}\frac{1}{h_1}\left(\frac{\partial p^{(0)}}{\partial \alpha_1} - h_1 M_1^{(0)}\right)\varepsilon^2 s(1-s) + U_{10}(1-s), \tag{39}$$

$$v_3^{(0)} = -\frac{1}{2\eta_0}\frac{1}{h_3}\left(\frac{\partial p^{(0)}}{\partial \alpha_3} - h_3 M_3^{(0)}\right)\varepsilon^2 s(1-s) + U_{30}(1-s). \tag{40}$$

We integrate the continuity equation (19) with respect to the variable α_2. Imposing the boundary condition (30) on the velocity component in the gap height direction, we obtain:

$$v_2^{(0)} = -\frac{1}{h_1}\int_0^{\alpha_2}\frac{\partial v_1^{(0)}}{\partial \alpha_1}d\alpha_2 - \frac{1}{h_1 h_3}\int_0^{\alpha_2}\frac{\partial}{\partial \alpha_3}\left(h_1 v_3^{(0)}\right)d\alpha_2. \tag{41}$$

Imposing boundary condition (31) for $i = 2$ on solution (41), we have:

$$\int_0^{\varepsilon}\frac{\partial v_1^{(0)}}{\partial \alpha_1}d\alpha_2 + \frac{1}{h_3}\int_0^{\varepsilon}\frac{\partial}{\partial \alpha_3}\left(h_1 v_3^{(0)}\right)d\alpha_2 = 0. \tag{42}$$

We insert solutions (39) and (40) into (42). Hence, the pressure $p_1^{(0)}$ for the steady conditions and magnetic field is determined by the following modified Reynolds equation:

$$\frac{1}{h_1}\frac{\partial}{\partial \alpha_1}\left[\frac{\varepsilon^3}{\eta_0}\left(\frac{\partial p^{(0)}}{\partial \alpha_1} - h_1 M_1^{(0)}\right)\right] + \frac{1}{h_3}\frac{\partial}{\partial \alpha_3}\left[\frac{h_1}{h_3}\frac{\varepsilon^3}{\eta_0}\left(\frac{\partial p^{(0)}}{\partial \alpha_3} - M_3^{(0)}\right)\right]$$

$$= 6U_{10}\frac{\partial \varepsilon}{\partial \alpha_1} + 6U_{30}\frac{1}{h_3}\frac{\partial[\varepsilon h_1]}{\partial \alpha_3}, \tag{43}$$

where $s \equiv \alpha_2/\varepsilon$, $0 \le r \equiv \alpha_2 \le \varepsilon$, $b_m \le \alpha_3 \equiv \vartheta \le b_s$.

5.2. Corrections for unsteady flow and viscoelastic properties

Imposing the boundary conditions (32), (34) on the system of equations (21)–(24) we obtain the changes of components of synovial fluid velocities v_1, v_3 caused by viscoelastic properties and unsteady motion in the following form:

$$v_i^{(k)} = \Pi_{ik} W_k + U_{ik} \frac{\sinh[(\varepsilon - \alpha_2) A_k]}{\sinh(\varepsilon A_k)} + V_{ik} \frac{\sinh(\alpha_2 A_k)}{\sinh(\varepsilon A_k)}, \tag{44}$$

where:

$$W_k \equiv [1 - \exp(\alpha_2 A_k)] - [1 - \exp(\varepsilon A_k)] \frac{\sinh(\alpha_2 A_k)}{\sinh(\varepsilon A_k)}, \tag{45}$$

$$\Pi_{ik} \equiv \frac{\mathbf{i}}{k\omega_0 \rho_0 h_i} \left(\frac{\partial p^{(k)}}{\partial \alpha_i} - \frac{1}{2^k} h_i M_i^{(k)} \right), \quad A_k \equiv \sqrt{\frac{\mathbf{i} k\omega_0 \rho_0}{\eta_k}} \tag{46}$$

for $i = 1, 3$ and $k = 1, 2, 3, \ldots$ Integrating continuity equation (24) with respect to the variable α_2 for boundary condition (33) we obtain:

$$v_2^{(k)} = -\frac{1}{h_1} \int_0^{\alpha_2} \frac{\partial v_1^{(k)}}{\partial \alpha_1} d\alpha_2 - \frac{1}{h_1 h_3} \int_0^{\alpha_2} \frac{\partial}{\partial \alpha_3} \left(h_1 v_3^{(k)} \right) d\alpha_2 . \tag{47}$$

Imposing the boundary condition (36) on the solution (47) we arrive at:

$$\int_0^{\varepsilon} \frac{\partial v_1^{(k)}}{\partial \alpha_1} d\alpha_2 + \frac{1}{h_3} \int_0^{\varepsilon} \frac{\partial}{\partial \alpha_3} \left(h_1 v_3^{(k)} \right) d\alpha_2 = -\mathbf{i}\, k\omega_0 \varepsilon^{(k)} h_1 . \tag{48}$$

If we take into account the rule of differentiation of the integrals with variable limits of integration and if we use additionally conditions (34), then equation (48) assumes the following form:

$$\frac{\partial}{\partial \alpha_1} \int_0^{\varepsilon} v_1^{(k)} d\alpha_2 + \frac{1}{h_3} \frac{\partial}{\partial \alpha_3} \int_0^{\varepsilon} h_1 v_3^{(k)} d\alpha_2 = -\mathbf{i}\, k\omega_0 \varepsilon^{(k)} h_1 + V_{1k} \frac{\partial \varepsilon}{\partial \alpha_1} + V_{3k} \frac{h_1}{h_3} \frac{\partial \varepsilon}{\partial \alpha_3} \tag{49}$$

for $k = 1, 2, 3, \ldots$ If we insert solutions (44) for $i = 1, 3$ into equation (49), then after final calculations in Appendix we obtain:

$$\frac{1}{h_1} \frac{\partial}{\partial \alpha_1} \left[\frac{\varepsilon^3}{\eta_k} \left(\frac{\partial p^{(k)}}{\partial \alpha_1} - \frac{1}{2^k} h_1 M_1^{(k)} \right) \right] + \frac{1}{h_3} \frac{\partial}{\partial \alpha_3} \left[\frac{h_1}{h_3} \frac{\varepsilon^3}{\eta_k} \left(\frac{\partial p^{(k)}}{\partial \alpha_3} - \frac{1}{2^k} h_3 M_3^{(k)} \right) \right]$$

$$= 12\mathbf{i}\, k\omega_0 h_1 \varepsilon^{(k)} - 12 \left[V_{1k} \frac{\partial \varepsilon}{\partial \alpha_1} + V_{3k} \frac{h_1}{h_3} \frac{\partial \varepsilon}{\partial \alpha_3} \right]$$

$$+6\left[U_{1k}(\alpha_3)+V_{1k}\right]\left[\frac{\partial\varepsilon}{\partial\alpha_1}-\frac{1}{12}\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3}{\eta_k}\right)\mathbf{i}k\omega_0\rho_0\right]$$

$$+6[U_{3k}(\alpha_1)+V_{3k}]\frac{1}{h_3}\left[\frac{\partial(\varepsilon h_1)}{\partial\alpha_3}-\frac{1}{12}\frac{\partial}{\partial\alpha_3}\left(\frac{\varepsilon^3 h_1}{\eta_k}\right)\mathbf{i}k\omega_0\rho_0\right] \tag{50}$$

for $k = 1, 2, 3, \ldots\ 0 \le \alpha_1 \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \le \varepsilon$.

Multiplying both sides of equation (50) by expression $\exp(\mathbf{i}k\omega_0 t)$ and equating terms of real parts of complex number on both sides of equation, we obtain the following sequence of the modified Reynolds equations:

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left[\frac{\varepsilon^3}{\eta_k^*}\left(\frac{\partial p^{(k)}}{\partial\alpha_1}-\frac{h_1}{2^k}M_1^{(k)}\right)\right]+\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left[\frac{h_1}{h_3}\frac{\varepsilon^3}{\eta_k^*}\left(\frac{\partial p^{(k)}}{\partial\alpha_3}-\frac{h_3}{2^k}M_3^{(k)}\right)\right]=-12\,k\omega_0 h_1\varepsilon^{(k)}\sin(k\omega_0 t)$$

$$+6\left[U_{1k}(\alpha_3)+V_{1k}\right]\left[\frac{\partial\varepsilon}{\partial\alpha_1}\cos(k\omega_0 t)+\frac{1}{12}\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3\eta_0}{\eta_0^2+\omega_0^2k^2\beta^2}\right)k\omega_0\rho_0\sin(k\omega_0 t)\right]$$

$$+6\left[U_{3k}(\alpha_1)+V_{3k}\right]\frac{1}{h_3}\left[\frac{\partial(\varepsilon h_1)}{\partial\alpha_3}\cos(k\omega_0 t)+\frac{1}{12}\frac{\partial}{\partial\alpha_3}\left(\frac{\varepsilon^3\eta_0 h_1}{\eta_0^2+\omega_0^2k^2\beta^2}\right)k\omega_0\rho_0\sin(k\omega_0 t)\right]$$

$$-12\left[V_{1k}\frac{\partial\varepsilon}{\partial\alpha_1}+V_{3k}\frac{h_1}{h_3}\frac{\partial\varepsilon}{\partial\alpha_3}\right]\cos(k\omega_0 t) \tag{51}$$

for $k = 1, 2, 3, \ldots\ 0 < \alpha_1 \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \vartheta \le \pi R/2 \equiv b_s$, $0 \le r \le \varepsilon$ and

$$\frac{1}{\eta_k^*}\equiv\Re\mathrm{e}\,\frac{\exp(\mathbf{i}k\omega_0 t)}{\eta_k}=\frac{\eta_0}{\eta_0^2+\omega_0^2k^2\beta^2}\cos(k\omega_0 t)+\frac{\omega_0 k\beta}{\eta_0^2+\omega_0^2k^2\beta^2}\sin(k\omega_0 t). \tag{52}$$

Formula (52) shows that total apparent viscosity η_k^* of synovial fluid depends additionally on the time t. This fact can be explained only by virtue of rheological properties of synovial fluid. The modified Reynolds equation (51) determines the following pressure functions: $p^{(1)}$, $p^{(2)}$, …, $p^{(k)}$. These functions define pressure corrections caused by the unsteady and viscoelastic properties of the synovial fluid in magnetic field.

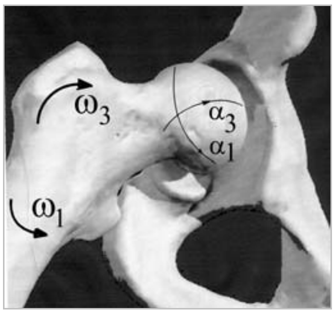
6. Particular cases of human joints

In a particular case of hip joint with spherical bone head, we have spherical coordinates and the Lamé coefficients in the following form:

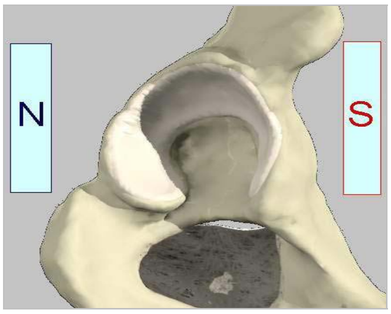
$$\alpha_1 \equiv \varphi, \quad \alpha_2 \equiv r, \quad \alpha_3 \equiv \vartheta, \quad h_1 = R\sin(\vartheta/R), \quad h_3 = 1. \tag{53}$$

The time-independent coefficients of circumferential velocities of spherical bone head can be expressed as (see figure 1):

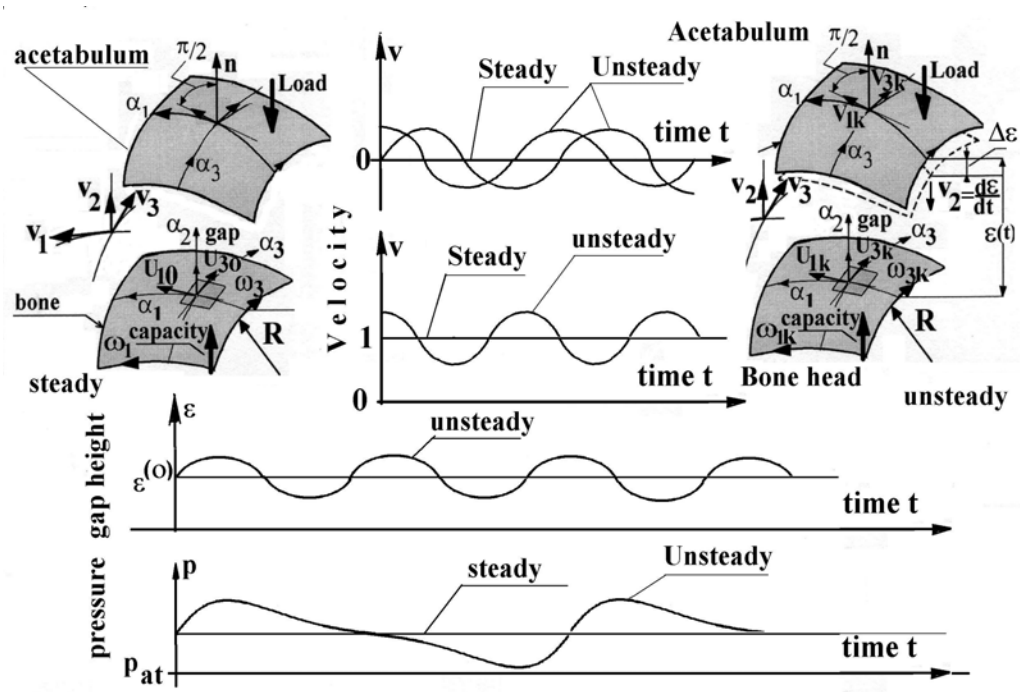
$$U_{10} = \omega_1 h_1 = \omega_1 R\sin(\vartheta/R), \; U_{30} = \omega_3 R\sin(\varphi), \; U_{1k} \equiv \omega_{1k} R\sin(\vartheta/R), \; U_{3k} \equiv \omega_{3k} R\sin\varphi\,,$$



a)



b)



c)

Fig. 1. Rotational periodic unsteady motion of bone head and acetabulum in two directions:
a, b – hip joint, c – bone head and acetabulum for arbitrary human joint in periodic motion, *n* – normal vector

$$\omega_{1k} \equiv \frac{\omega_{10}}{k^2}, \quad \omega_{3k} \equiv \frac{\omega_{30}}{k^2}, \quad k = 1, 2, 3, ..., \tag{54}$$

where ω_1, ω_{10} are angular velocities of spherical bone head and their perturbations in circumferential direction ($\alpha_1 = \varphi$), and ω_3, ω_{30} are angular velocities of spherical bone head and their perturbations in meridional direction ($\alpha_3 = \vartheta$). Symbol R denotes in this case the radius of spherical bone head.

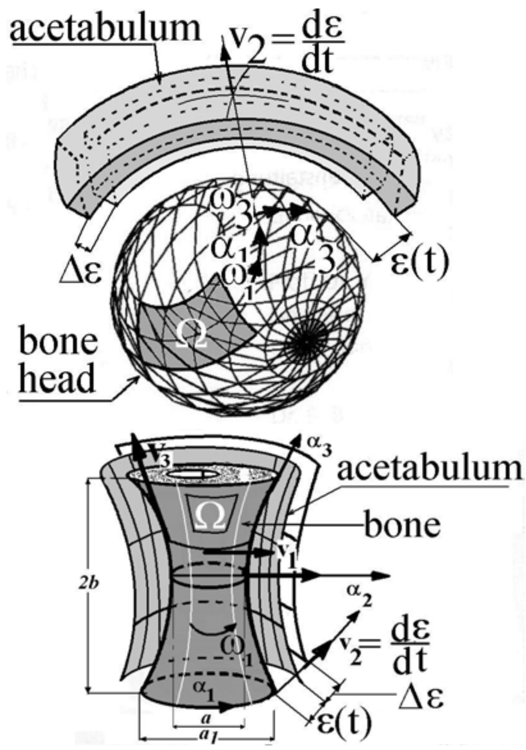


Fig. 2. Range of the region of lubrication on the spherical and hyperbolic bone heads

In hyperbolical coordinates (α_1, α_2, α_3), for hyperbolic bone head in human joint the following Lamé coefficients are valid:

$$h_1 = a\sec^2(\alpha_3\Lambda), \quad h_2 = 1, \quad h_3 = a\sec^2(\alpha_3\Lambda)\sqrt{1 + 4(a\Lambda)^2\tan^2(\alpha_3\Lambda)} \tag{55}$$

with

$$\Lambda \equiv \frac{1}{b}\sqrt{\frac{w}{a}}, \quad 0 \le \alpha_1 \le 2\pi, \quad 0 \le \alpha_2 \le \varepsilon, \quad |\alpha_3| \le \frac{1}{\Lambda}\arccos\sqrt{\frac{a}{a+w}}.$$

We make the following notations: a is the smallest radius of the bone cross-section, $a_1 = a + w$ is the largest radius of the bone cross-section, $w \equiv a_1 - a$, $2b$ is the joint length.

The region of lubrication $\Omega(\alpha_1, \alpha_3)$ (a bone head) of spherical hip joint in spherical co-ordinates and the region of lubrication $\Omega(\alpha_1, \alpha_3)$ (an acetabulum) of hyperbolic hip joint in hyperbolic coordinates under unsteady conditions are shown in figure 2.

7. Modified Reynolds equations in spherical co-ordinates

Let us present the modified Reynolds equation for unsteady motion in magnetic field, but without viscoelastic properties of synovial fluid, i.e., for $\beta = 0$. We assume that a centre of spherical bone head is at the point $O(0,0,0)$ and centre of spherical cartilage at the point $O_1(x - \Delta\varepsilon_1, y - \Delta\varepsilon_2, z + \Delta\varepsilon_3)$. Eccentricity has the following value: $D \equiv [(\Delta\varepsilon_1)^2 + (\Delta\varepsilon_2)^2 + (\Delta\varepsilon_3)^2]^{0.5}$ (see figure 3). In spherical coordinates, we assume thin boundary layer, thus for synovial fluid flow we obtain: $\alpha_1 = \varphi$, $\alpha_3 = \vartheta$ and the Lamé coefficients (53). We also assume time-independent coefficients of gap height perturbations in the form: $\varepsilon^{(k)} \equiv \varepsilon^{(0)}/k^2$, hence by virtue of (37) an average gap height is a sum of infinite series in the following form [3]:

$$\varepsilon \equiv \varepsilon^{(0)} + \varepsilon^{(0)}\sum_{k=1}^{\infty}\frac{\sin(\omega_0 k t_m)}{k^3\omega_0 t_m}, \quad \varepsilon = \Gamma_0\varepsilon^{(0)}, \quad \Gamma_0 \equiv 1 + 0.08333\,(2\pi - \omega_0 t_m)\,(\pi - \omega_0 t_m), \tag{56}$$

where:

$$\begin{aligned}\varepsilon^{(0)}(\varphi, \vartheta/R) \equiv{}& \Delta\varepsilon_1\cos\varphi\sin(\vartheta/R) + \Delta\varepsilon_2\sin\varphi\sin(\vartheta/R) - \Delta\varepsilon_3\cos(\vartheta/R) - R \\ &+ \{[\Delta\varepsilon_1\cos\varphi\cdot\sin(\vartheta/R) + \Delta\varepsilon_2\sin\varphi\sin(\vartheta/R) - \Delta\varepsilon_3\cos(\vartheta/R)]^2 \\ &+ (R + \varepsilon_{min})(R + 2D + \varepsilon_{min})\}^{0.5},\end{aligned} \tag{57}$$

$$\sum_{k=1}^{\infty}\frac{\sin(kx)}{k^3 x} = \frac{(2\pi - x)\,(\pi - x)}{12} \quad \text{for} \quad 0 < x \le 2\pi. \tag{58}$$

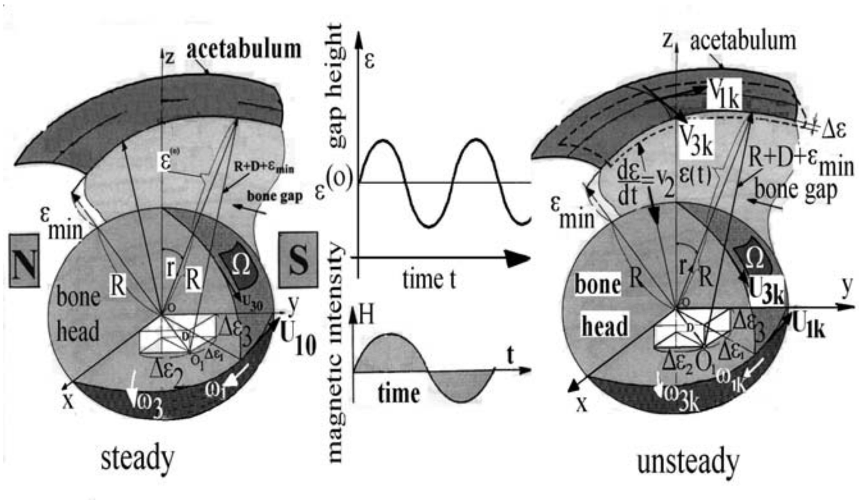


Fig. 3. Time-dependent changes of the gap height under boundary conditions

If left and right sides of equations (43) and (51) are added for $k = 1, 2, 3, \ldots$, and viscoelastic properties of synovial fluid are neglected, i.e. the coefficient β tends to zero, then we arrive at the modified Reynolds equation:

$$\frac{1}{R\sin\frac{\vartheta}{R}}\frac{\partial}{\partial\varphi}\left\{\frac{\varepsilon^3}{\eta_0}\left[\frac{\partial p}{\partial\varphi}-R\left(\sin\frac{\vartheta}{R}\right)\sum_{k=0}^{\infty}\frac{1}{2^k}M_1^{(k)}\cos(k\omega_0 t)\right]\right\}$$

$$+R\frac{\partial}{\partial\vartheta}\left\{\frac{\varepsilon^3}{\eta_0}\left[\frac{\partial p}{\partial\vartheta}-\sum_{k=0}^{\infty}\frac{1}{2^k}M_3^{(k)}\cos(k\omega_0 t)\right]\sin\frac{\vartheta}{R}\right\}$$

$$=6\omega_1 R\left(\sin\frac{\vartheta}{R}\right)\frac{\partial\varepsilon}{\partial\varphi}+6\omega_3 R^2(\sin\varphi)\frac{\partial}{\partial\vartheta}\left(\varepsilon\sin\frac{\vartheta}{R}\right)$$

$$-12\,\omega_0 R\left(\sin\frac{\vartheta}{R}\right)\varepsilon^{(0)}\sum_{k=1}^{\infty}\frac{\sin(\mathrm{k}\omega_0 t)}{k}-12\left[V_{10}\frac{\partial\varepsilon}{\partial\varphi}+R\sin\left(\frac{\vartheta}{R}\right)V_{30}\frac{\partial\varepsilon}{\partial\vartheta}\right]\sum_{k=1}^{\infty}\frac{\cos(k\omega_0 t)}{k^2}$$

$$+6\left[\omega_{10}R\left(\sin\frac{\vartheta}{R}\right)+V_{10}\right]\left[\frac{\partial\varepsilon}{\partial\varphi}\sum_{k=1}^{\infty}\frac{\cos(k\omega_0 t)}{k^2}+\frac{1}{12}\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3}{\eta_0}\right)\omega_0\rho_0\sum_{k=1}^{\infty}\frac{\sin(k\omega_0 t)}{k}\right]$$

$$+6R[\omega_{30}R(\sin\varphi)+V_{30}]$$

$$\times\left[\frac{\partial}{\partial\vartheta}\left(\varepsilon\sin\frac{\vartheta}{R}\right)\sum_{k=1}^{\infty}\frac{\cos(\mathrm{k}\omega_0 t)}{k^2}+\frac{1}{12}\frac{\partial}{\partial\vartheta}\left(\frac{\varepsilon^3}{\eta_0}\sin\frac{\vartheta}{R}\right)\omega_0\rho_0\sum_{k=1}^{\infty}\frac{\sin(\mathrm{k}\omega_0 t)}{k}\right], \quad (59)$$

where $M_1 \equiv M_\varphi$, $M_3 \equiv M_\vartheta$ and the sums of infinite series assume the following forms [3]:

$$\sum_{k=1}^{\infty} \frac{\sin(k\omega_0 t)}{k} \equiv \frac{\pi - \omega_0 t}{2} \quad \text{for} \quad 0 < \omega_0 t < 2\pi,$$

$$\sum_{k=1}^{\infty} \frac{\cos(k\omega_0 t)}{k^2} \equiv \frac{(\pi - \omega_0 t)^2}{4} - \frac{\pi^2}{12} \quad \text{for} \quad 0 \le \omega_0 t \le 2\pi .$$

We have $0 \le \varphi \le \pi$, $\pi R/8 \le \vartheta \le \pi R/2$ and the gap height has the form $\varepsilon = \Gamma_0\, \varepsilon^{(0)}$, whereas $\Gamma_0 = 1$ for $\omega_0 t_m = 2\pi$, and $\Gamma_0 = 1 + 0.1273\pi^2$ for $\omega_0 t_m = \pi/6$. The modified Reynolds equation (59) determines the total pressure function p for unsteady conditions in magnetic field.

8. Numerical calculations

The pressure distribution $p^{(0)}$ for stationary motion and its corrections $p^{(1)}$, $p^{(2)}$, $p^{(3)}$, ... for unsteady flow of synovial fluid with viscoelastic properties can be calculated from equations (43), (51) in the lubrication region Ω indicated in figure 3. It is a section of the bowl of the sphere. In this case, on the boundary of the region Ω the pressure $p^{(0)}$ takes the value of atmospheric pressure p_{at} and corrections of pressures $p^{(k)}$ for $k = 1, 2$ are equal to zero. We use the time-depended viscosity (52).

Total pressure p for the unsteady flow of synovial fluid in hip joint gap without viscoelastic properties we determined inside the region Ω from equation (59). On the boundary of the region Ω the total pressure p assumes the values of atmospheric pressure p_{at}. Numerical calculations are performed for the region Ω: $0 \le \varphi \le \pi$, $\pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2$, where $\Gamma_0 = 1$ for $\omega_0 t_m = 2\pi$, and $\varepsilon = \Gamma_0\, \varepsilon^{(0)}$, $R = 0.0265$ [m], $\omega_1 = 0.8$ [s^{-1}], $\omega_3 = 0.150$ [s^{-1}], $\omega_0 = 0.02$ [s^{-1}], $\omega_{10} = 0.09$ [s], $\omega_{30} = 0.01$ [s], $\Delta\varepsilon_1 = 1$ [μm], $\Delta\varepsilon_2 = 0.5$ [μm], $\Delta\varepsilon_3 = 3$ [μm], $\eta_0 = 0.20$ [Pas], $\rho_0 = 800$ [kg/m^3]. Minimal value of the gap height $\varepsilon_{\min} = 3.0$ [μm], and maximal value of the gap height $\varepsilon_{\max} = 7.12$ [μm] and we take into account the time period $t = 2\pi/\omega_0$. Numerical calculations of pressure distributions varying with time are presented in figures 4, 5 and 6.

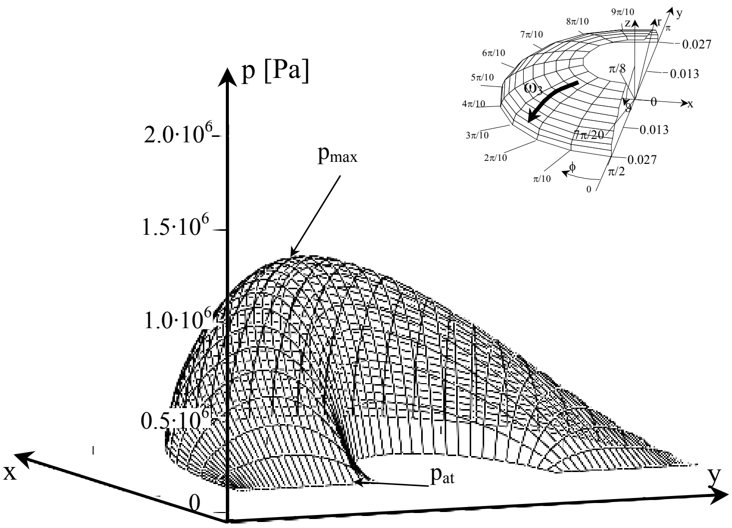
Figure 4 shows pressure distribution varying with time caused by rotation ($\omega_3 = 0.15$ s^{-1}) of the bone head in the meridional direction ($\alpha_3 \equiv \vartheta$) only for normal hip joint which is not affected by magnetic field. We take into account the angular velocity perturbations ($\omega_{30} = 0.010$ s^{-1}) on the spherical head of the bone in meridional directions, i.e., $\omega_1 = 0$, $\omega_{10} = 0$. Rotation about the bone head in circumferential direction is not taken into account. We assume that acetabulum is motionless, i.e., $V_{ik} = 0$. We also assume the perturbations of the joint gap height in unsteady motion at angular velocity ω_0 equal to 0.02 s^{-1}. For the time $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, … we obtain the maximal values of pressure, which are $1.067\cdot10^6$ Pa; $1.245\cdot10^6$ Pa; $1.067\cdot10^6$ Pa, …, respectively.

Figure 5 shows pressure distribution varying with time caused by rotation (ω_1 = 0.8 s^{-1}) of the bone head in the circumferential direction ($\alpha_1 \equiv \varphi$) for normal hip joint which is not affected by magnetic field. We taken into account the angular velocity perturbations (ω_{10} = 0.09 s^{-1}) on the spherical head of the bone in circumferential directions. Rotation about the bone head in meridional direction is not taken into account, i.e. $\omega_3 = 0$, $\omega_{30} = 0$. Acetabulum is motionless, i.e. $V_{ik} = 0$. For the sake of a better comparison of numerical results, we assume the perturbations of the same gap height at angular velocity ω_0 = 0.02 s^{-1}. For the times $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, … we obtain the maximal values of pressure, which are 1.186·10^6 Pa; 0.932·10^6 Pa; 1.186·10^6 Pa, …, respectively.

Figure 6 shows pressure distribution varying with time caused by rotation (ω_1 = 0.8 s^{-1}) of the bone head in the circumferential direction ($\alpha_1 \equiv \varphi$) and simultaneously by rotation (ω_3 = 0.15 s^{-1}) of the bone head in meridional direction ($\alpha_3 \equiv \vartheta$) for normal hip joint being not affected by magnetic field. We take into account the angular velocity perturbations (ω_{10} = 0.09 s^{-1}) on the spherical head in circumferential direction ($\alpha_1 \equiv \varphi$) and simultaneously velocity perturbations (ω_{30} = 0.010 s^{-1}) on the spherical head of the bone in meridional directions. We assume the same perturbations of gap height in unsteady motion at angular velocity ω_0 = 0.02 s^{-1}. For the times $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, … we obtain maximal values of pressure equal to 2.077·10^6 Pa; 2.002·10^6 Pa; 2.077·10^6 Pa, …, respectively.

R=0.0265 [m], η=0.20 [Pas]
ω_1=0.0 [1/s], ω_{10}=0.0 [1/s]
ω_3=0.15 [1/s], ω_{30}=0.01 [1/s]
ω_0=0.02 [1/s]

t=0 [s] and t=2·π/ω_o [s]
p_{max}=1.067·10^6 [Pa]
C_{tot}=859.6 [N]
Lubrication surface =20.38



R=0.0265 [m], η=0.20 [Pas]
ω_1=0.0 [1/s], ω_{10}=0.0 [1/s]
ω_3=0.15 [1/s], ω_{30}=0.01 [1/s]
ω_0=0.02 [1/s]

t=π/ω_o [s]
p_{max}=1.245·10^6 [Pa]
C_{tot}=1012 [N]
Lubrication surface =20.38 [cm^2]

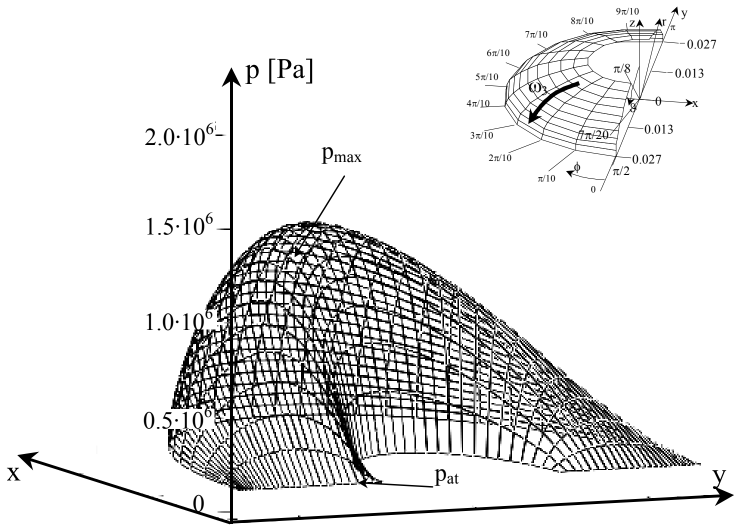
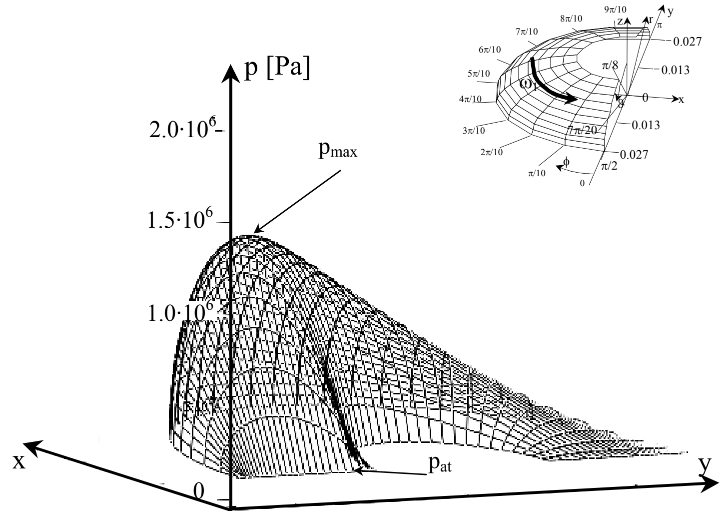


Fig. 4. Pressure distributions caused by rotation in $\alpha_3 \equiv \vartheta$ direction only ($\omega_1 = 0$, $\omega_{10} = 0$), where non-zero values of angular velocity $\omega_3 = 0.15$ s^{-1}, and non-zero angular velocity perturbations ω_{30} in unsteady flow and non-zero angular velocity perturbations ω_0 of gap height perturbations are taken into account

R=0.0265 [m], η=0.20 [Pas]
ω_1=0.8 [1/s], ω_{10}=0.09 [1/s]
ω_3=0.0 [1/s], ω_{30}=0.00 [1/s]
ω_0=0.02 [1/s]

t=0 [s] and t=2π/ω_o [s]
p_{max}=1.186·10^6 [Pa]
C_{tot}=865.4 [N]
Lubrication surface =20.38 [cm^2]



R=0.0265 [m], η=0.20 [Pas]
ω_1=0.8 [1/s], ω_{10}=0.09 [1/s]
ω_3=0.0 [1/s], ω_{30}=0.00 [1/s]
ω_0=0.02 [1/s]

t=π/ω_o [s]
p_{max}=0.932·10^6 [Pa]
C_{tot}=670 [N]
Lubrication surface =20.38 [cm^2]

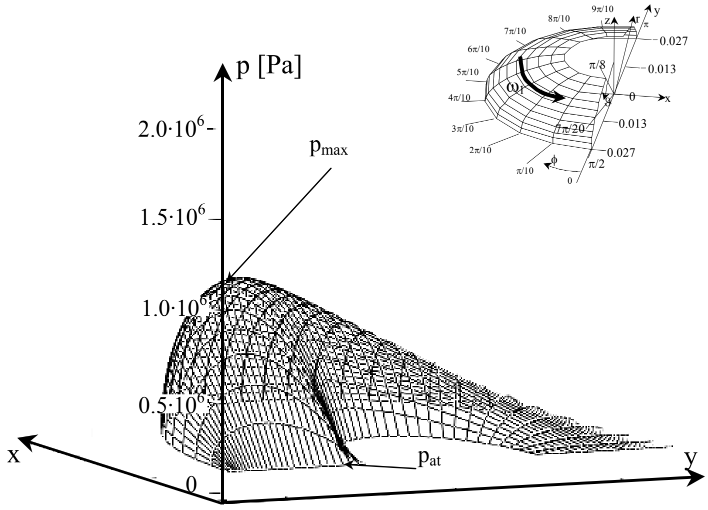
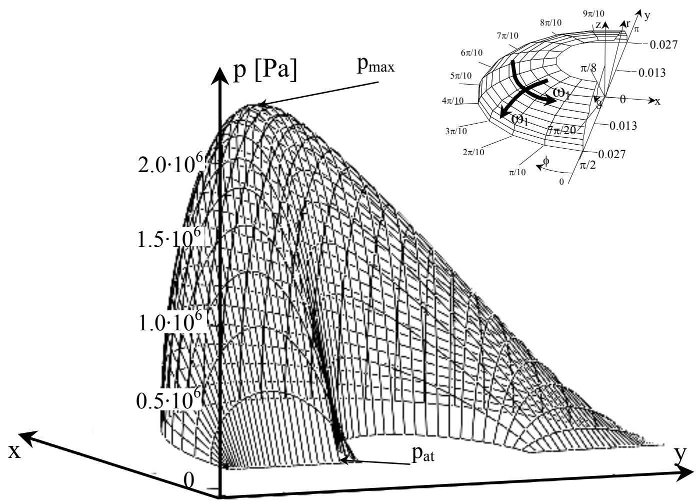


Fig. 5. Pressure distributions caused by rotation in $\alpha_1 \equiv \varphi$ direction only ($\omega_3 = 0$, $\omega_{30} = 0$), where non-zero values of angular velocity $\omega_1 = 0.8$ s^{-1}, and non-zero angular velocity perturbations ω_{10} in unsteady flow and non-zero angular velocity perturbations ω_0 of gap height perturbations are taken into account

R=0.0265 [m], η=0.20 [Pas]
ω_1=0.8 [1/s], ω_{10}=0.09 [1/s]
ω_3=0.15 [1/s], ω_{30}=0.01 [1/s]
ω_0=0.02 [1/s]

t=0 and t=2π/ω_o [s]
p_{max}=2.077·10^6 [Pa]
C_{tot}=1696 [N]
Lubrication surface =20.38 [cm^2]



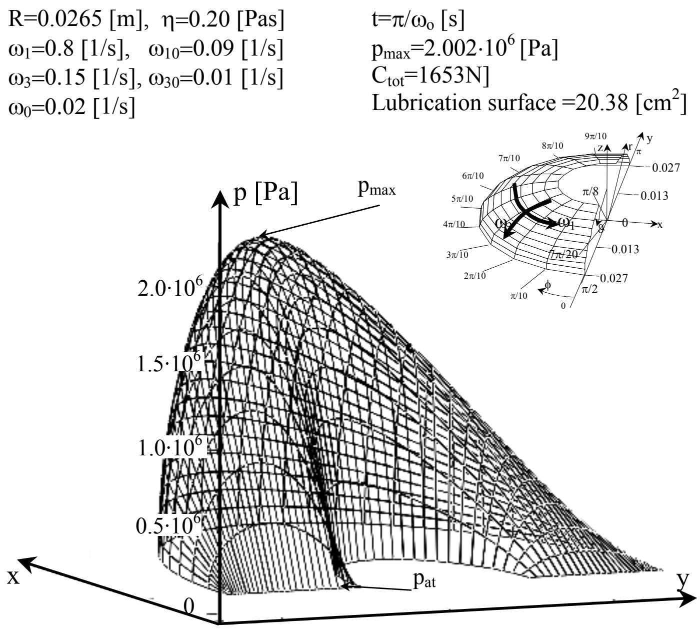


Fig. 6. Pressure distributions caused by rotation of bone head in circumferential direction $\alpha_1 \equiv \varphi$ and simultaneously in meridional direction $\alpha_3 \equiv \vartheta$, where non-zero angular velocities $\omega_1 = 0.8\ \mathrm{s}^{-1}$, $\omega_3 = 0.15\ \mathrm{s}^{-1}$ and non-zero angular velocity perturbations ω_{10}, ω_{30} in unsteady flow and non-zero angular velocity perturbations ω_0 of gap height perturbations are taken into account. Symbol $C_{\mathrm{tot.}}$ denotes a total pressure

The first pictures in figures 4, 5 and 6 show the pressure distributions for initial and final times of the period of perturbations of the motion of human joint. The second pictures in figures 4, 5 and 6 present the pressure distributions for middle time point of the period of perturbations of the motion. Afterwards the pressure distributions return to the distributions, which are shown in the first pictures of figures 4, 5 and 6.

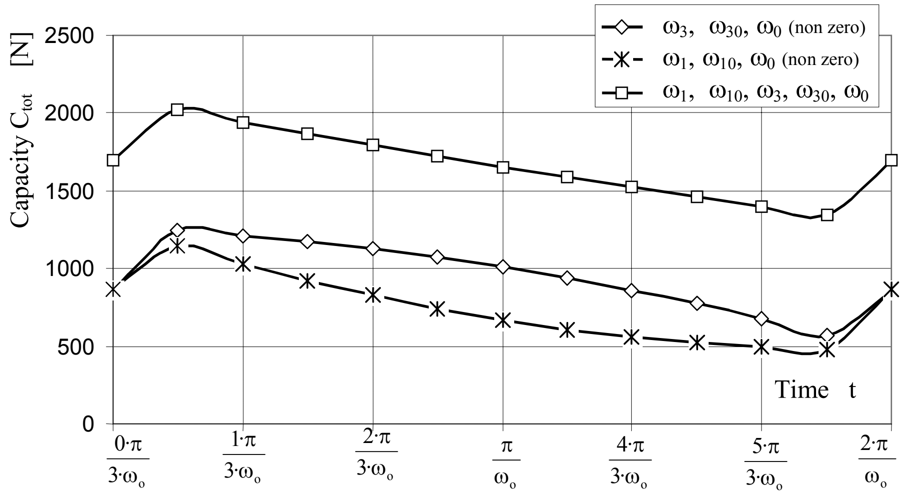


Fig. 7. Capacity distributions versus time for three assumptions corresponding to the three cases presented in figures 4, 5 and 6, i.e. for the motion of bone head in meridional direction ◊, circumferential direction ×, and simultaneously in circumferential and meridional directions ∇, respectively

Figure 7 presents three curves of total capacity distributions versus time in the range of time of the perturbation period. Calculations are performed for the following times: $t = 0$ [s], $t = \pi/3\omega_0$ [s], $t = 2\pi/3\omega_0$ [s], $t = \pi/\omega_0$ [s], $t = 4\pi/3\omega_0$ [s], $t = 5\pi/3\omega_0$ [s], $t = 2\pi/\omega_0$ [s], … For the motion of bone head in meridional direction and for the time $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, $t = 3\pi/\omega_0$, $t = 4\pi/\omega_0$ we obtain the following values of capacities: 859 N, 1012 N, 859N, 1012 N, 859 N, respectively. For the motion of bone head in circumferential direction and for the times $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, $t = 3\pi/\omega_0$, $t = 4\pi/\omega_0$ we obtain the following values of capacities: 865 N, 670 N, 865 N, 670 N, 865 N, respectively. For simultaneous motion of bone head in circumferential and meridional directions and for the times $t = 0$, $t = \pi/\omega_0$, $t = 2\pi/\omega_0$, $t = 3\pi/\omega_0$, $t = 4\pi/\omega_0$ we obtain the following values of capacities: 1696 N, 1653 N, 1696 N, 1653 N, 1696 N, respectively.

It is easy to see that the pressure distributions and capacities for the times: $t = 0$ [s], $t = 2\pi/\omega_0$ [s], $t = 4\pi/\omega_0$ [s] have the same values. The pressure distributions and capacities for the time $t = (k - 1)\pi/\omega_0$ [s] at $k = 2, 3, 4, ...$ have the same values as well.

9. Conclusions

In the present paper, analytical and numerical solutions of the pressure and velocities of synovial fluid for any human joint in curvilinear orthogonal coordinates are presented. Periodic perturbations of unsteady lubrication and simultaneously of viscoelastic properties of the fluid in magnetic field are taken into account. In numerical calculations, done for the pressure and capacity distributions, only perturbations of the motion of human hip joint surfaces are included.

A new form of the Reynolds equation derived in this paper tends in particular case to a well-known form of the Reynolds equation for steady motion being derived in earlier papers. The results obtained reveal that the total apparent viscosity of synovial fluid depends on the time and on the velocity deformations. Total apparent viscosity of synovial fluid changes periodically in time.

An unsteady magnetic induction field equal to 0.1 mT with the frequency of about 60 Hz changes pressure distribution in human hip joint from 1 to 4 per cent.

Appendix

If we substitute solutions (44) for (45), (46) in equation (49), then we have:

$$\frac{\partial}{\partial\alpha_1}\left[\frac{1}{h_1}\frac{\partial p^{(k)}}{\partial\alpha_1}\int_0^{\varepsilon}W_k d\alpha_2\right]+\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left[\frac{h_1}{h_3}\frac{\partial p^{(k)}}{\partial\alpha_3}\int_0^{\varepsilon}W_k d\alpha_2\right]$$

$$+\frac{\partial}{\partial\alpha_1}\left\{\frac{k\omega_0\rho_0}{\mathbf{i}}\left[U_{1k}\int_0^{\varepsilon}\frac{\sinh[(\varepsilon-\alpha_2)A_k]}{\sinh[\varepsilon A_k]}d\alpha_2+V_{1k}\int_0^{\varepsilon}\frac{\sinh[\alpha_2 A_k]}{\sinh(\varepsilon A_k)}d\alpha_2\right]\right\}$$

$$+\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left\{\frac{k\omega_0\rho_0}{\mathbf{i}}h_1\left[U_{3k}\int_0^{\varepsilon}\frac{\sinh[(\varepsilon-\alpha_2)A_k]}{\sinh[\varepsilon A_k]}d\alpha_2+V_{3k}\int_0^{\varepsilon}\frac{\sinh(\alpha_2 A_k)}{\sinh[\varepsilon A_k]}d\alpha_2\right]\right\}$$

$$=-k^2\omega_0^2\rho_0\varepsilon^{(k)}h_1+\frac{\mathrm{k}\omega_0\rho_0}{\mathbf{i}}\left[V_{1k}\frac{\partial\varepsilon}{\partial\alpha_1}+V_{3k}\frac{h_1}{h_3}\frac{\partial\varepsilon}{\partial\alpha_3}\right] \tag{A1}$$

for $k = 1, 2, 3, \ldots\ 0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \equiv r$.

For further reduction of equation (A1) it is necessary to calculate the following integrals:

$$\frac{k\omega_0\rho_0}{\mathbf{i}}U_{ik}\int_0^{\varepsilon}\frac{\sinh[(\varepsilon-\alpha_2)A_k]}{\sinh[\varepsilon A_k]}d\alpha_2=\frac{k\omega_0\rho_0}{\mathbf{i}}U_{ik}\int_0^{\varepsilon}\frac{e^{(\varepsilon-\alpha_2)A_k}-e^{-(\varepsilon-\alpha_2)A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}d\alpha_2$$

$$=\frac{k\omega_0\rho_0}{\mathbf{i}A_k}U_{ik}\frac{e^{\varepsilon A_k}-2+e^{-\varepsilon A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}=\frac{k\omega_0\rho_0}{\mathbf{i}A_k}U_{ik}\tanh\left(\frac{\varepsilon A_k}{2}\right)=-\frac{1}{2}\mathbf{i}k\omega_0\rho_0U_{ik}\left[\varepsilon-\frac{1}{12}\varepsilon^3A_k^2+O\left(\varepsilon^4\right)\right], \tag{A2}$$

$$\int_0^{\varepsilon}W_k d\alpha_2=\int_0^{\varepsilon}\left[\left(1-e^{\alpha_2A_k}\right)-\left(1-e^{\varepsilon A_k}\right)\frac{e^{\alpha_2A_k}-e^{-\alpha_2A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}\right]d\alpha_2=\int_0^{\varepsilon}\left[1-\frac{e^{\alpha_2A_k}-e^{(\varepsilon-\alpha_2)A_k}}{1+e^{\varepsilon A_k}}\right]d\alpha_2$$

$$=\varepsilon+\frac{2}{A_k}\frac{1-e^{\varepsilon A_k}}{1+e^{\varepsilon A_k}}=\varepsilon+\frac{2}{A_k}\tanh\left(\frac{\varepsilon A_k}{2}\right)=\frac{\mathbf{i}}{12}\frac{\varepsilon^3}{\eta_{pk}}k\omega_0\rho_0-O\left(\varepsilon^4\right), \tag{A3}$$

$$V_{ik}\frac{k\omega_0\rho_0}{\mathbf{i}}\int_0^{\varepsilon}\frac{\sinh\alpha_2 A_k}{\sinh\varepsilon A_k}d\alpha_2 = V_{ik}\frac{k\omega_0\rho_0}{\mathbf{i}}\int_0^{\varepsilon}\frac{e^{\alpha_2 A_k}-e^{-\alpha_2 A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}d\alpha_2$$

$$=\frac{k\omega_0\rho_0}{\mathbf{i}\,A_k}V_{ik}\frac{e^{\varepsilon A_k}-2+e^{-\varepsilon A_k}}{e^{\varepsilon A_k}-e^{-\varepsilon A_k}}=\frac{k\omega_0\rho_0}{\mathbf{i}\,A_k}V_{ik}\tanh\left(\frac{\varepsilon A_k}{2}\right)=-\frac{1}{2}\mathbf{i}\,k\omega_0\rho_0 V_{ik}\left[\varepsilon-\frac{1}{12}\varepsilon^3 A_k^2+O\left(\varepsilon^4\right)\right]. \quad \text{(A4)}$$

Integrals (A2), (A3), (A4) are inserted into equation (A1). Thus we arrive at equation (50).

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