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Original paper

Minimum-time running: a numerical approach

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The article deals with the minimum-time running problem. The time of covering a given distance is minimized. The Hill–Keller model of running employed is based on Newton's second law and the equation of power balance. The problem is formulated in optimal control. The unknown function is the runner's velocity that varies with the distance. The problem is solved applying the direct Chebyshev's pseudospectral method.

Key words: *minimum-time running, Chebyshev's pseudospectral method*

1. Introduction

The basic question relating to a competitive running is: how should a runner vary his speed with the distance to minimize the time during which he covers a given distance? This problem may be considered using the formalism of optimal control. In the problem under consideration, the papers of KELLER [4], [5] are of a fundamental importance for running on the flat track. Keller formulates the equivalent problem, i.e., the maximization of the distance of the run for a given time. This problem is solved equating the first variation of the functional to zero. The simple model of runner's motion is employed. The resisting force exerted on a competitor linearly depends on his velocity. Inequality constraints are imposed onto a propulsive force and on the energy released during athlete's metabolism. The optimum velocity consists of three arcs: the acceleration, the cruise with a constant velocity and the "negative kick" at the end. The same problem (maximization of the distance for a given time) is considered by BEHNCKE [1]. His model of athlete's motion is more general – it includes the resisting force, depending on the velocity squared, therefore the reasoning may be extended to swimming. He employs Pontryagin's maximum principle. The optimal control depends on the adjoint variables, and they have not a clear physical interpretation. The proof given in his paper is very sophisticated and his assumptions are sometimes unclear. COOPER [2] cities the problem formulation, the model of athlete's motion and the method presented by BEHNCKE [1]. A new element is the set of experimental data for the wheelchair races. WOODSIDE [16], based on earlier results, assumes that the optimal solution consists of three velocity arcs: an acceleration during which a runner uses his maximal propulsive force, "cruising" with a constant velocity, and deceleration when the energy reserves are depleted. His attention is focused on computing the switching times separating these three intervals. MAROŃSKI [8] considers the original problem – minimization of the time for a given-distance run. He employs the method of Miele (extremization of linear integrals using Green's theorem). In his model, the resisting force may be a quadratic function of the velocity, therefore he extends the results to swimming. This algorithm given by MAROŃSKI [7] is relatively simple and therefore the reasoning is

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recalled by TÖZEREN [15]. He confirms the earlier results that the optimum velocity consists of three segments with a constant velocity in the middle. This result agrees with the conclusion of SANDERSON and MARTINDALE [14] referring to optimal rowing technique. However, Maroński's approach has a drawback – it may be applied to relatively simple models of running.

All the methods mentioned above are based on analytical considerations. They employ the necessary conditions of the optimality known from the calculus of variations or the optimal control (PINCH [10]). These methods, called indirect methods, are not very efficient in numerical applications. They may generate the so-called wild solutions. This is an inherent feature in the two-point boundary value problem of ordinary differential equations (ROBERTS and SHIPMAN [12]). In direct numerical methods, the state and the control variables versus time are represented by polynomials. The coefficients of these polynomials are optimized, not the functions (like in the calculus of variations). In such a way, the optimal control problem is converted to a nonlinear programing problem (NLP). The authors of this paper have experimented with both kinds of numerical methods (MAROŃSKI and ŁUCJANEK [6], PANASZ and MAROŃSKI [9], ROGOWSKI and MAROŃSKI [13]). The direct methods seem to be much more promising. That is why the minimum-time running problem is reconsidered in this paper.

Formulation of the problem

Let us consider the Hill–Keller model of running (PRITCHARD [11]). Its fundamental assumptions are as follows:

1. The racer is regarded as a particle that coincides with his centre of gravity. This means that his body dimensions are small compared to the distance to be covered. Vertical displacements associated with the cyclic nature of the stride pattern and at the start of the race are neglected.

2. The motion occurs in a vertical plane. The velocity is not reduced in turnings.

3. The strategy is irrespective of the decisions of other competitors. The runner is isolated from the group.

4. The goal to be achieved is to cover a given distance in minimum time.

5. The runner's velocity is changed via the variations of the propulsive force setting. The greater propulsive force causes the greater losses of competitor's energy.

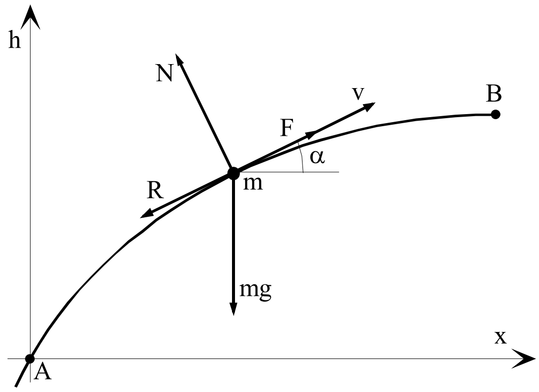

Fig. 1. Forces exerted on the competitor. F is the propulsive force varying with the distance, $F = mf_{\max}(v)\eta$; R is the resisting force, $R = mr(v)$; N – the normal reaction of the ground; mg – the weighing force; α – the local slope angle; v – the velocity

Applying Newton's second law, after projecting the forces onto the tangent line to the path (figure 1), the following equation is obtained:

$$\frac{dv}{dx} = \frac{f_{\max}(v)\eta - r(v)}{v}, \tag{1}$$

where: v is runner's velocity, x – the distance covered (independent variable), $f_{\max}(v)$ – maximal propulsive force per unit mass, η – propulsive force setting varying with the distance, $r(v)$ – resistance per unit mass.

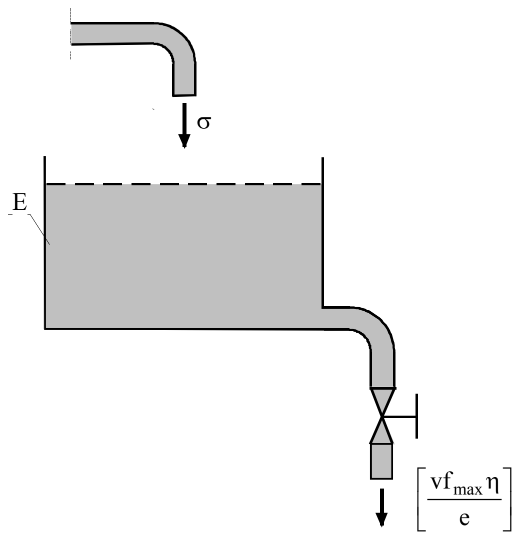

Fig. 2. Hydraulic analogy of the energy conversion in competitor's body. The description in the text

The energy conversion in the competitor's body may be interpreted using a simple hydraulic analogy (figure 2). The competitor's chemical energy E is represented by the volume of the fluid in a vessel. The volumetric rate on the input refers to the recovery rate σ. The volumetric rate on the output is controlled by the competitor (a tap in figure 2) and it is proportional to the power used at a given instant of time. The vessel is filled up with the fluid at the beginning and may be emptied during the process, but the amount of the fluid cannot be negative. The equation of power balance is as follows:

$$\frac{dE}{dx} = \frac{\sigma(v)}{v} - \frac{f_{\max}(v)\eta}{e(v)}, \quad (2)$$

where: E denotes actual reserves of chemical energy per unit mass in excess of the non-running metabolism, $\sigma(v)$ – recovery rate of chemical energy per unit mass, $e(v)$ – the efficiency of converting the chemical energy into mechanical one. The propulsive force setting η (the control variable) is not known a priori. It should be taken from the given range

$$0 \le \eta \le 1. \quad (3)$$

The upper limit refers to maximal propulsive force that cannot be exceeded. The amount of energy E cannot be negative during the race

$$E(x) \ge 0. \quad (4)$$

Boundary conditions should supplement the state equations (1) and (2).

The problem is formulated as follows: Find $v(x)$, $\eta(x)$ and $E(x)$ satisfying (1)÷(4) so that the time T of covering a given distance is minimized

$$T = \int_0^D \frac{1}{v}\, dx \Rightarrow \text{MIN}. \quad (5)$$

Symbol D in the upper limit of functional (5) denotes the distance to be covered.

2. Method

In the paper, the method of FAHROO and ROSS [3] is applied. It employs the N-th degree Lagrange polynomial approximations for the state (v, E) and the control variable (η) with the values of these variables at the Chebyshev–Gauss–Lobatto (CGL) points as the expansion coefficients. The state and control variables at the CGL points are the unknown parameters to be optimized in nonlinear programing problem (NLP). To solve NLP problem the function *fmincon* from *Optimization Toolbox* of MATLAB is used. *Fmincon* is an implementation of the sequential quadratic programing method. It allows mimicking Newton's method for constrained optimization. At each major iteration an approximation of the Hessian of the Lagrangian function is made using a quasi-Newton updating method and then it is used to determine a search direction for a line search procedure.

Example

Let us consider the following problem as an example. The competitor should cover the distance of 400 m in the minimum time. The track is inclined to the horizontal. The inclination angle is constant. We will employ the Hill–Keller model of competitive running in which the resisting force linearly depends on the velocity (KELLER [4], [5], PRITCHARD [11]):

$$r(v) = \frac{v}{\tau} + g\sin\alpha, \quad (6)$$

where τ is the constant dumping coefficient, g is the acceleration due to gravity, α is the local slope angle. The data are taken from MAROŃSKI [8]. The optimum velocity diagram versus the distance covered is given in figure 3.

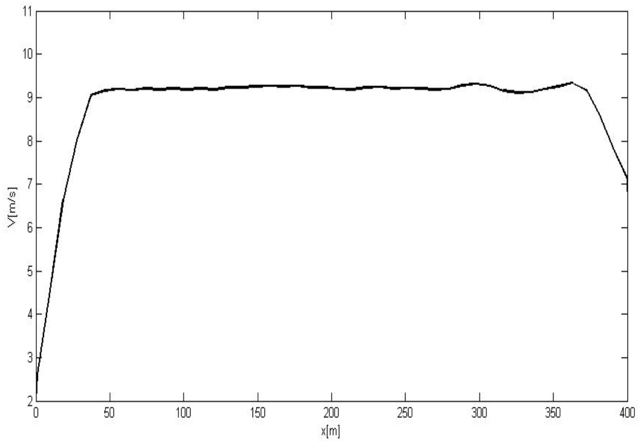

Fig. 3. The optimum velocity v versus the distance x covered in the 400-m run

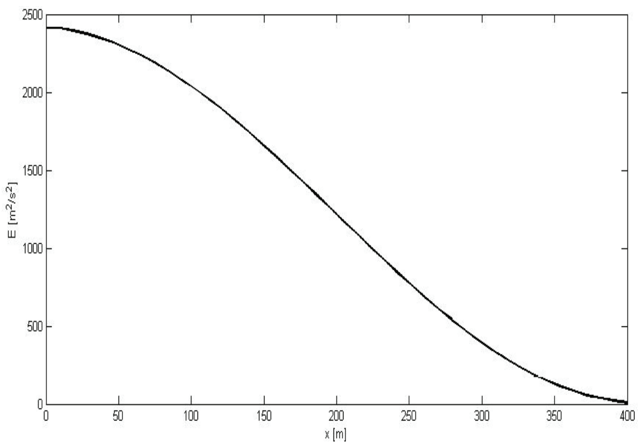

Fig. 4. The actual reserves of chemical energy per unit mass E versus the distance x covered in the 400-m run

3. Disscusion and summary

Computing the time of covering a given distance is possible via the integration of Newton's equation of motion (1) for the assumed propulsive force setting η (PRITCHARD [11]). For middle- and long-distance events such an approach does not give unique results. Different competitor's strategies, represented by different velocity profiles, are responsible for different times of the run. That is why the problem should be formulated and solved using optimal control. And here two general methods may be applied: indirect one based on the necessary conditions of optimality following from equating the first variation of the functional to zero and direct one, where the problem of optimal control is converted to NLP problem. In this paper, the direct Chebyshev's pseudospectral method based on the paper by FAHROO and ROSS [3] is employed.

Our findings confirm the earlier results of KELLER [4], [5] and MAROŃSKI [8]. In the races, whose distances exceed some critical value, the optimal strategy is to accelerate from the very start, applying maximum propulsive force until a speed is achieved that is maintained throughout the race. This velocity is approximately constant. After depleting all the energy resources the velocity decreases at the finish. The formulation of the problem is very flexible in the present approach. Any inequality constraints imposed on the control and the state variables may be considered. The number of state and control variables may be greater than that in Miele's method where the problem should be reduced to extremization of a line integral in the (E, v)-plane, which is not always possible (MAROŃSKI [8]). However, the method has a drawback since, it is moderately stable. The results are acceptable for a relatively large number of node points.

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