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Capacity of deformed human joint gap in time-dependent magnetic field

KRZYSZTOF WIERZCHOLSKI

Base Technique Department, Maritime Academy of Gdynia, PL-81-225 Gdynia, Morska str 83.
Tel.:058- 690-13-48, E-mail: wierzch@wsm.gdynia.pl

Analysis of carrying capacity of synovial unsymmetrical fluid flow in deformed, human joint gap, especially in hip joint, is presented. The following assumptions are taken into account: stationary, isothermal and incompressible synovial unsymmetrical fluid flow in time-dependent magnetic field, rotational motion of bone head, squeeze of synovial fluid in human joint gap, changeable synovial non-Newtonian fluid viscosity, changeable and deformed gap height in human joint, and constant synovial fluid density.

The simplified system of basic equations for pressure and synovial velocity distribution are analysed.

Numerical and analytical formulae for capacity force taking into account conjugation fields of the stresses and deformations occurring in elastic cartilage and in synovial fluid obtained by virtue of theory of elasticity and fluid mechanics can be considered as the novely of this paper. Analytical solutions for the values of capacity forces allow easy numerical calculations, which may be very useful for medical diagnosis.

Key words: hip joint, non-Newtonian synovial fluid, unsteady magnetic field, deformed cartilage

1. Introduction

In the papers mentioned in references, their authors discuss not only hydrodynamic parameters (i.e. synovial fluid velocity components, pressure in joint gap), but also mechanical parameters (i.e. friction forces, friction coefficients, capacity). They use as a rule both analytical and numerical methods. A multitude of performed methods of solutions is shown in table 1.

Papers [18], [20], [24]–[28] present an idea of friction forces in various human joints for various geometry of bone co-operating surfaces, for changeable joint gap height and for unsymmetrical flow of synovial fluid in magnetic field. In calculations of friction, we took into account the velocity components in circumference and longitudinal directions [28].

Table. Short review of the papers dealing with working parameters in human joint

Paper	Synovial fluid flow		Hydrodynamic lubrication due to		Magnetic field	Working parameters obtained analytically (a) and numerically (n)
	Symmetrical	Unsymmetrical	Squeezing	Rotation		
[1]	yes	no	yes	yes	no	pressure (a), friction forces (a), friction coefficient
[8]	yes	no	yes	no	no	friction forces
[12]	not defined	not defined	not defined	not defined	yes	lubrication ability
[13]	yes	no	no	yes	no	pressure, capacity (a), (n)
[15]	yes	no	no	yes	no	friction force, capacity friction coefficient (a), (n)
[16]	yes	no	no	yes	no	pressure distribution capacity, (a), (n)
[17]	yes	yes	no	yes	no	total solutions (a), (n)
[18]	yes	no	no	yes	no	friction force (a)
[19]	yes	yes	no	yes	yes	friction force (a)
[20]	yes	yes	no	yes	yes	friction force (a)
[21]	yes	yes	yes	yes	yes	pressure distribution (a)
[22]	yes	yes	no	yes	yes	pressure distribution (a)
[23]	yes	no	yes	no	no	pressure distribution (a)
[24]	yes	no	yes	no	no	friction force (a)
[25]	no	yes	yes	no	no	pressure distribution (a), synovial fluid velocity (a)
[27]	no	yes	yes	no	no	pressure distribution (a)

The problem of capacity force in human joint gap for changeable height of joint gap and for unsymmetrical flow of synovial fluid in magnetic field has not been discussed in papers [19]–[22], [25], [27], [28]. The novelty of the present paper is carrying the capacity calculations of human hip joint for deformed bone and cartilage surfaces lubricated due to unsymmetrical flow in curvilinear, orthogonal co-ordinates and in the presence of magnetic unsteady field.

2. Main assumptions

The following assumptions are accepted:

- Rotational motion of the head of hip bone.
- Squeeze motion of bone head.
- Unsymmetrical flow of synovial fluid.
- Stationary and isothermal flow of fluid.
- Constant density of synovial fluid.
- Changeable and deformed dynamic viscosity of synovial fluid.
- Changeable gap height of human joint gap.
- Changeable magnetic induction field.

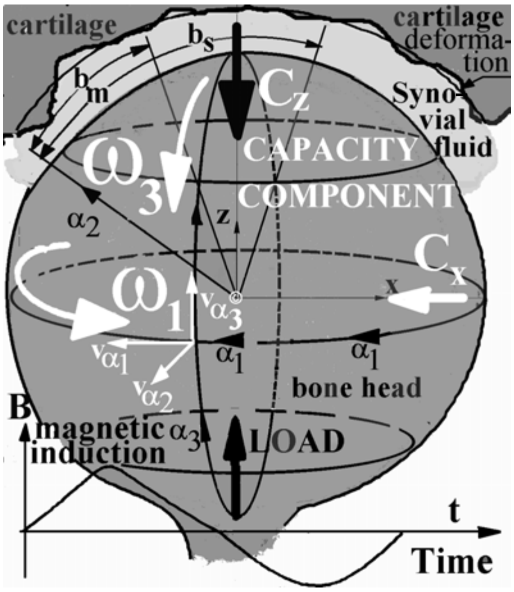


Fig. 1. Capacity force C_z in human hip joint in spherical coordinates for hydrodynamic lubrication with rotation as an opposite reaction to loading force (gap in enlarged scale)

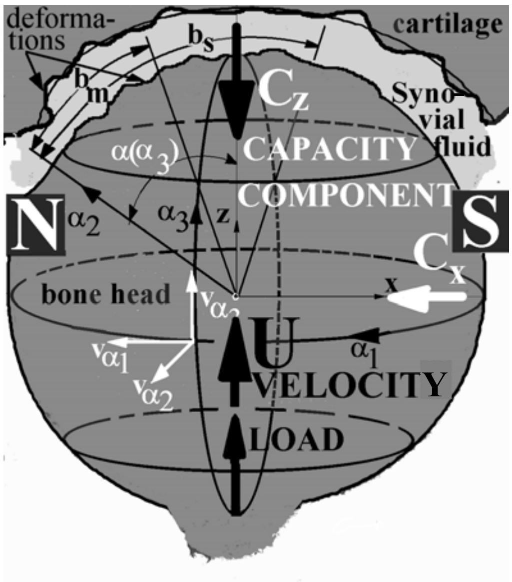


Fig. 2. Capacity force C_z in human hip joint in spherical coordinates for hydrodynamic lubrication with squeezing as an opposite reaction to loading force (gap in enlarged scale)

The changes of gap height may be generated by the geometry of a head of bone, namely by some irregularities caused by local deformations of cartilage and by roughness of bone surface. Elastohydrodynamic effects are considered in normal human joints because in the case of large athletic efforts occurring in some sports or in the case of pathological joints we may find some deformations of cartilage surface due to hydrodynamic pressure. We take into account a roughness of bone surface and pathological irregularities of bone surface caused by various diseases, because they contribute greatly to the gap height.

A spherical bone surface in hip joint creates a curvilinear spherical joint gap. Figure 1 shows the capacity force C_z in human hip joint in the case of its hydrodynamic lubrication caused by rotation motion of head of bone in circumference or meridian direction. The values of magnetic induction field are changed periodically. Figure 2 shows the capacity force C_z in human hip joint in the case of hydrodynamic lubrication due to squeeze motion of bone head in indicated direction. The arbitrary rotational bone surfaces create curvilinear joint gap filled with a synovial fluid. The motion of bone causes the flow of synovial fluid.

Figure 2 presents two co-operating bone surfaces during squeeze lubrication of human joints with synovial fluid in magnetic induction field. Two curvilinear bone surfaces separated by joint gap of small height come up at the uniform velocity U. This velocity is caused by motion of human limbs. Figure 1 shows two various co-operating bone surfaces during their lubrication with synovial fluid due to the bone head rotation in magnetic induction field. Rotation motion of head of bone at an angular velocity ω and radius R is caused by motion of human limbs.

Relations between dynamic viscosity of synovial fluid and shear rate are presented in figure 3. In figure 3a, there are tested bullock's ankle and knee joint fluids in a Weissenberg rheogoniometer in the rotation mode. In figure 3b and figure 3c, numerical and experimental values for synovial fluid reported in [2] are presented. At low shear rates the values of the coefficient of apparent viscosity are constant and the fluid has Newtonian characteristics. At high shear rates they are shear-thinning. Theoretical formulation of constitutive equations for the synovial fluid is reviewed by LAI, KUEI and MOW [2]. The viscosity of synovial fluid of non-Newtonian properties was examined experimentally by DOWSON [1], MOW et al. [7] and MOW and GUILAK [9]. Using numerical values obtained by WIERZCHOLSKI, PYTKO [13] and WIERZCHOLSKI et al. [14] we arrive at approximation formulae for dynamic viscosity values for small and large shear rates:

$$\eta_p \equiv \eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + A \cdot \Theta} \approx \eta_0 - (\eta_0 - \eta_\infty)\Theta A + \ldots \quad \text{for} \quad 0 < \Theta^2 B << 1,$$

$$\eta_p \equiv \eta_\infty + \frac{\eta_0 - \eta_\infty}{1 + A \cdot \Theta + B \cdot \Theta} \approx \eta_0 - (\eta_0 - \eta_\infty)\Theta A - (\eta_0 - \eta_\infty)\Theta^2 B + \ldots \quad \text{for} \quad \Theta^2 B \geq 1, \tag{1}$$

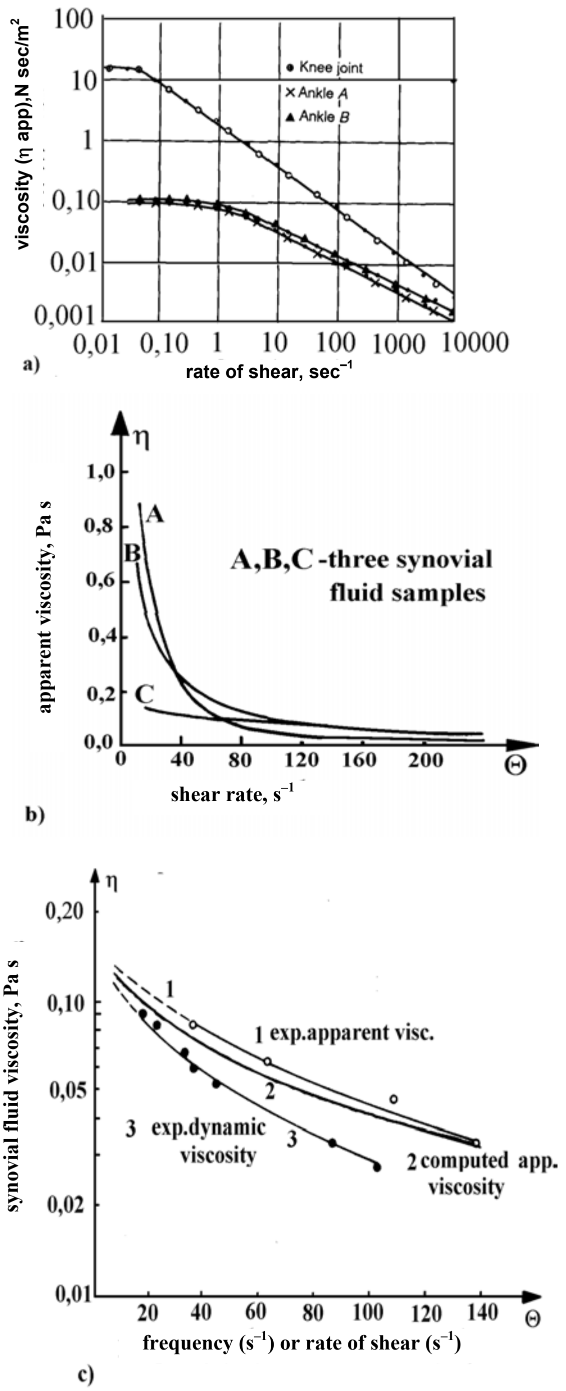


Fig. 3. Dynamic viscosity of synovial fluid versus shear rate

where η_∞ stands for dynamic viscosity of synovial fluid at large shear rate, η_0 is characteristic dynamic viscosity of synovial fluid (in Pa·s) at small shear rate. The symbols A and B denoting empirical coefficients obtained by Dowson depend additionally on magnetic induction field. The coefficients obtained numerically acquire the following values: A = 1.88307 s and B = 0.00458 s^2 for normal human joint and also A = 0.03349 s and B = 0.00131 s^2 for pathological human joint if magnetic field does not appear. The shear rate has the following form:

$$\Theta_0 \cong O\left(\frac{V_0}{\varepsilon}\right), \quad \Theta \equiv \frac{\partial V_1}{\partial \alpha_2}. \tag{2}$$

3. The Helmholtz equations for unsteady electromagnetic field

Maxwell equations in unsteady electromagnetic field are as follows [10]:

$$\mathrm{rot}\,\mathbf{H} = \sigma\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t}, \quad \mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\mathbf{B} = 0, \quad \mathrm{div}\mathbf{D} = \rho_e, \quad \mathrm{div}(\mathrm{rot}\,\mathbf{H}) = \mathrm{div}\left(\sigma\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0, \tag{3}$$

where: $\mathbf{D} = \mu_e\mathbf{E}$, $\mathbf{B} = \mathbf{H}\mu$, ρ_e – electric charge of space in synovial fluid (As/m^3), $\mathbf{D}$ – electric induction vector (As/m^2), $\mathbf{E}$ – electric intensity vector (mkgs^{-3}A), σ – coefficient of electrical conductivity of synovial fluid (s^{3}A^{2}m^{-3}kg), H_i – components of the vector $\mathbf{H}$ of magnetic intensity (A/m), $B_i = \mu H_i$ – components of the vector $\mathbf{B}$ of magnetic induction (T), $N_i = \chi H_i$ – components of magnetisation vector $\mathbf{N}$ (A/m), μ – coefficient of magnetic permeability of synovial fluid (mkgs^{-2}A^{-2}), μ_e – coefficient of electric permeability of synovial fluid (s^{4}A^{2}m^{-3}kg), χ – dimensionless magnetisation intensity. If we deal with homogeneous and isotropic synovial fluid without electric charge of space and at $\rho_e = 0$, then equations (3) have the following form:

$$\mathrm{rot}\,\mathbf{H} = \sigma\,\mathbf{E} + \mu_e\frac{\partial \mathbf{E}}{\partial t}, \quad \mathrm{rot}\,\mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}, \quad \mathrm{div}\,\mathbf{H} = 0, \quad \mathrm{div}\,\mathbf{E} = 0, \quad \left(\sigma + \mu_e\frac{\partial}{\partial t}\right)\mathrm{div}\,\mathbf{E} = 0. \tag{4}$$

If the following identity

$$\mathrm{rot}\,(\mathrm{rot}\,\mathbf{R}) = \mathrm{grad}\,(\mathrm{div}\,\mathbf{R}) - \nabla^2\mathbf{R} \tag{5}$$

is taken into account, then equations (4) will tend to the following partial differential hyperbolic equations:

$$\nabla^2\mathbf{H} = \mu\sigma\,\frac{\partial \mathbf{H}}{\partial t} + \mu\mu_e\frac{\partial^2\mathbf{H}}{\partial t^2}, \quad \nabla^2\mathbf{E} = \mu\sigma\,\frac{\partial\,\mathbf{E}}{\partial t} + \mu\mu_e\frac{\partial^2\mathbf{E}}{\partial t^2}. \tag{6}$$

If synovial fluid is a good insulator, i.e. $\sigma = 0$, then from (6) it will follow:

$$\nabla^2 \mathbf{H} = \mu\mu_e \frac{\partial^2 \mathbf{H}}{\partial t^2}, \quad \nabla^2 \mathbf{E} = \mu\mu_e \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{7}$$

For synovial fluid that conducts electric current, i.e. for $\sigma >> \mu_e \approx 0$, from (6) we obtain the following Helmholtz equations:

$$\nabla^2 \mathbf{H} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t}, \quad \nabla^2 \mathbf{E} = \mu\sigma \frac{\partial \mathbf{E}}{\partial t}. \tag{8}$$

4. Basic equations and sketch of solutions

Lubrication of human hip joint will be described by means of equations of conservation of momentum and equation of continuity for steady motion of synovial fluid in thin gap and unsteady magnetic field. Moreover, we take into account the equilibrium equations for thin layer of cartilage. We neglect centrifugal forces because of small velocities and terms of the order of $Re\,\Psi$, $\Psi \equiv \varepsilon / R \approx 10^{-3}$, where R stands for the radius of curvature of bone surface, and ε for the gap height. Boundary simplifications of the system of conservation of momentum, continuity and Maxwell equations for synovial fluid in a thin gap as well as simplifications of the equations of elasticity for bone and cartilage (see Appendix 1) allows us to arrive at the system of the following form[7]–[11]:

$$0 = -\frac{1}{h_1}\frac{\partial p}{\partial \alpha_1} + \frac{\partial}{\partial \alpha_2}\left(\eta_p \frac{\partial v_1}{\partial \alpha_2}\right) + \frac{N_1}{h_1}\frac{\partial B_1}{\partial \alpha_1} + \frac{N_3}{h_3}\frac{\partial B_1}{\partial \alpha_3}, \tag{9}$$

$$0 = \frac{\partial p}{\partial \alpha_2}, \tag{10}$$

$$0 = -\frac{1}{h_3}\frac{\partial p}{\partial \alpha_3} + \frac{\partial}{\partial \alpha_2}\left(\eta_p \frac{\partial v_3}{\partial \alpha_2}\right) + \frac{N_1}{h_1}\frac{\partial B_3}{\partial \alpha_1} + \frac{N_3}{h_3}\frac{\partial B_3}{\partial \alpha_3}, \tag{11}$$

$$h_3 \frac{\partial v_1}{\partial \alpha_1} + h_1 h_3 \frac{\partial v_2}{\partial \alpha_2} + \frac{\partial}{\partial \alpha_3}(h_1 v_3) = 0, \tag{12}$$

$$\frac{1}{h_1 h_3}\left[\frac{\partial}{\partial \alpha_1}\left(\frac{h_3}{h_1}\frac{\partial H_i}{\partial \alpha_1}\right) + \frac{\partial}{\partial \alpha_3}\left(\frac{h_1}{h_3}\frac{\partial H_i}{\partial \alpha_3}\right)\right] = \mu \in \frac{\partial^2 H_i}{\partial t^2} \quad \text{for} \quad i = 1, 3, \tag{13}$$

$$\frac{\partial}{\partial \alpha_2}\left[\left(2G + \Lambda\delta_{ij}\right)\frac{\partial u_i}{\partial \alpha_2}\right] = \delta_{2i}\left[\frac{\partial}{\partial \alpha_2}\left(3\, K \alpha_T T^*\right)\right] \quad \text{for} \quad i, j = 1, 2, 3, \tag{14}$$

where h_1, h_3 are the Lamé coefficients and $0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$, $0 \le \alpha_2 \equiv r \le \varepsilon$. We introduce the following denotations: $G \equiv 0.5E(1 + \nu)^{-1}$ stands for shear modulus, $\Lambda \equiv E\nu(1 + \nu)^{-1}(1 - 2\nu)^{-1}$, $K \equiv \Lambda + (2/3)G$ is the coefficient of cubic elastic, E is Young's modulus of the cartilage or bone, ν – the Poisson ratio of the cartilage or bone, α_T – thermal coefficient of linear expansion for cartilage or bone, T^* – temperature in cartilage, δ_{ij} – the Kronecker symbol equalled to unity for $i = j$ and zero for $i \ne j$. We assume curvilinear orthogonal α_1, α_2, α_3 coordinates in circumference, gap height and length directions, respectively. The symbols u_1, u_2, u_3 denote the components of displacement vector of cartilage body in circumference, gap height and meridian directions, respectively. For axially asymmetrical flow of synovial fluid, three components v_1, v_2, v_3 of its velocity vector depend on the variables α_1, α_2 and α_3, the pressure function depends on α_1, α_3, and dynamic viscosity η_p of synovial fluid depends on α_1, α_2 and α_3 . The gap height ε depends on the components u_i, hence it may be a function of the variables α_1 and α_3. Stresses and deformations occurring in elastic layer of cartilage and bone and in synovial fluid create conjugation fields present in the system of equations (9)–(14), which has been obtained by virtue of the theory of elasticity and fluid mechanics. Without loss of the generality, for the velocity components and the pressure, the following approach has been introduced:

$$v_i = v_i^{(0)}(\alpha_1, \alpha_2, \alpha_3) + Av_i^{(1)} + \ldots + A^k v_i^{(1)} + \ldots, \qquad i = 1, 2, 3,$$

$$p = p^{(0)}(\alpha_1, \alpha_2) + Ap^{(1)} + \ldots + A^k p^{(k)} + \ldots . \tag{15}$$

Symbol p denotes total pressure, symbol $p^{(0)}$ is the pressure for dynamic viscosity of synovial fluid independent of shear rate, and h_1, h_3 are the Lamé coefficients. For hip joint in spherical coordinates we have $h_1 \equiv R \sin(\alpha_3/R)$. Symbol $p^{(j)}$ for $j = 1, 2, 3, \ldots$ denotes the decrease or increase in the pressure and symbol $v^{(j)}$ for $j = 1, 2, 3,$ – velocities of synovial fluid caused by its non-Newtonian properties; in such a case viscosity depends on shear rate.

For hydrodynamic lubrication by means of squeezing, we impose classical boundary conditions [23], [24], [25] on the velocity component of synovial fluid in gap height direction, which makes it possible to obtain modified Reynolds equations for hydrodynamic pressure function $p(\alpha_1, \alpha_3)$ in the following form [10],[11]:

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left[\frac{\varepsilon^3(u_2)}{\eta_0}\left(\frac{\partial p^{(0)}}{\partial\alpha_1} - M_1(t)h_1\right)\right] + \frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left[\frac{h_1\varepsilon^3(u_2)}{h_3\eta_0}\left(\frac{\partial p^{(0)}}{\partial\alpha_3} - M_3(t)h_3\right)\right] = -12Uh_1, \tag{16}$$

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3(u_2)}{\eta_0}\frac{\partial p^{(1)}}{\partial\alpha_1}\right) + \frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left(\frac{h_1\varepsilon^3(u_2)}{h_3\eta_0}\frac{\partial p^{(1)}}{\partial\alpha_3}\right) = 0, \tag{17}$$

where $0 < \alpha_1 \le 2\pi$, $0 \le \alpha_3 \le R\pi c_3$, $0 \le \alpha_2 \equiv r \le \varepsilon$, $c_3 \in [0, 1/27]$ and

$$M_i \equiv (\mathbf{N}\nabla)B_i + 0.5\,\mathrm{rot}(\mathbf{N}\times\mathbf{B})_i\,.\; p = p^{(0)} + Ap^{(1)} + O(A^2). \tag{18}$$

In equations (16) and (17), the unknown functions $p^{(1)}$ and the term $A\,p^{(1)}$ occur. These functions describe the changes of pressure caused by the decrease in dynamic viscosity of synovial fluid due to the increase in the shear rate.

We impose a classical boundary condition [17], [18], [19], [20] on the velocity component of synovial fluid and especially on the component v_2 in gap height direction for hydrodynamic lubrication caused by bone rotation, hence we derive the modified Reynolds equations for hydrodynamic pressure function $p(\alpha_1, \alpha_3)$ in the following form [26]:

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left[\frac{\varepsilon^3(u_2)}{\eta_0}\left(\frac{\partial p^{(0)}}{\partial\alpha_1} - M_1(t)h_1\right)\right] + \frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left[\frac{h_1\varepsilon^3(u_2)}{h_3\eta_0}\left(\frac{\partial p^{(0)}}{\partial\alpha_3} - M_3(t)h_3\right)\right] = 6\omega h_1\frac{\partial\varepsilon(u_2)}{\partial\alpha_1}, \tag{19}$$

$$\frac{1}{h_1}\frac{\partial}{\partial\alpha_1}\left(\frac{\varepsilon^3(u_2)}{\eta_0}\frac{\partial p^{(1)}}{\partial\alpha_1}\right) + \frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left(\frac{h_1\varepsilon^3(u_2)}{h_3\eta_0}\frac{\partial p^{(1)}}{\partial\alpha_3}\right)$$
$$= -\frac{1}{2}\omega\frac{\partial}{\partial\alpha_1}\left[\frac{\varepsilon^2(u_2)\kappa_1}{\eta_\infty}\left(\frac{\partial p^{(0)}}{\partial\alpha_1} - M_1(t)h_1\right)\right]$$
$$-\frac{1}{4}\omega\frac{1}{h_3}\frac{\partial}{\partial\alpha_3}\left[\frac{h_1^2\varepsilon^2(u_2)\kappa_1}{\eta_\infty h_3}\left(\frac{\partial p^{(0)}}{\partial\alpha_3} - M_3(t)h_3\right)\right], \tag{20}$$

where $p \equiv p^{(0)} + Ap^{(1)} + O(A^2)$, $0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$ $\varepsilon = \varepsilon[u_2(p)]$, $-0.02 \le \kappa \equiv 4[(\eta_\infty)^2 - \eta_0\eta_\infty](\eta_\infty)^{-2} \le -0.04$.

If we insert the Duhamel–Neumann and strain–displacement relations [10] in boundary conditions (A2), (A7) for normal stress and pressure occurring in joint cartilage, we will obtain the following equation (see Appendix 1):

$$\frac{\partial u_2}{\partial\alpha_2}(\alpha_1, \alpha_2 = R+\varepsilon, \alpha_3) = \frac{3\alpha_T\kappa^*\Delta T(\alpha_2 = R+\varepsilon) - p}{2G+\Lambda}, \tag{21}$$

where: $u_2 = u_2(p, T)$ is the component of displacement vector in cartilage in gap height direction, ΔT is the function of temperature difference in cartilage layer in gap height direction obtained from heat transfer equation, and κ^* is the thermal conductivity of the cartilage body. Symbol p denotes total pressure. Equation (19) determines the unknown pressure function $p^{(0)}$ as the first approximation of total pressure, while equation (20) – the unknown pressure function $p^{(1)}$, i.e. $Ap^{(1)}$, which describes correction values of total pressure.

Integrating twice equation (14) for i = 2 with respect to the variable α_2 and assuming the boundary condition (21), (A7), we obtain elastic layer displacements u_i, where the unknown u_2 denotes displacement of cartilage in gap height direction (see Appendix 2).

5. Cartilage deformations in the region of lubrication

For oil film resting on spherical bone surface we have the following Lamé coefficients:

$$h_1 = R\sin(\alpha_3/R), \quad h_2 = 1, \quad h_3 = 1\,, \tag{22}$$

where R is the radius of a sphere. We introduce the following denotations: $\alpha_1 \equiv \phi$ – circumference direction, $\alpha_2 = r$ – gap height direction, and $\alpha_3 \equiv \vartheta$ – (meridian) direction. The dependencies between rectangular (x, y, z) and spherical ($\alpha_1 = \varphi$, $\alpha_2 = r$, $\alpha_3 = \vartheta$) co-ordinates have the following classical form:

$$x = r\sin\left(\frac{\vartheta}{R}\right)\cos\varphi, \quad y = r\sin\left(\frac{\vartheta}{R}\right)\sin\varphi, \quad z = r\cos\left(\frac{\vartheta}{R}\right), \quad 0 < r < R\,. \tag{23}$$

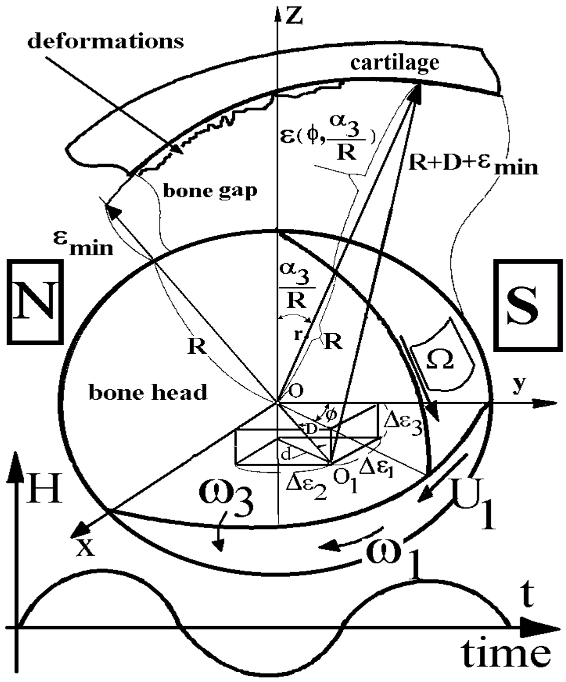


Fig. 4. Centers of spherical cartilage body and spherical bone head during lubrication caused by rotation in variable, time-dependent magnetic intensity *H* and deformed cartilage

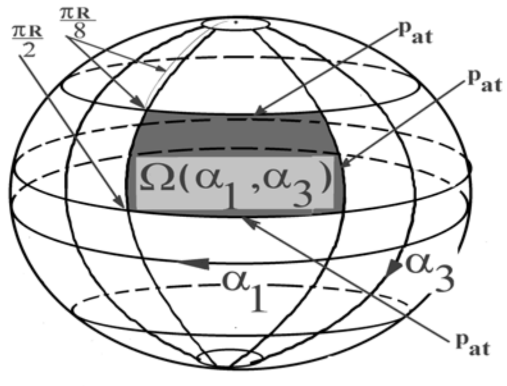


Fig. 5. Pressure-distribution region resting on surface of bone head during lubrication caused by rotation

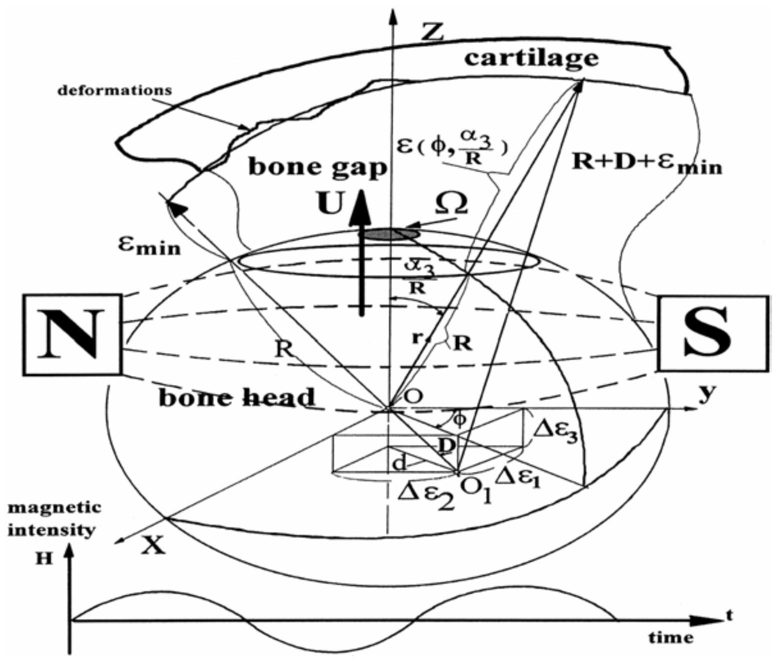


Fig. 6. Centers of spherical cartilage body and spherical bone head during lubrication caused by squeezing

The centre of a spherical bone head $O(0, 0, 0)$ and the centre of a spherical cartilage in the point $O_1(x - \Delta\varepsilon_1, y - \Delta\varepsilon_2, z + \Delta\varepsilon)$ for hydrodynamic lubrication caused by rotation is presented in figure 4, and for hydrodynamic lubrication caused by squeezing – in figure 6. A region of hydrodynamic lubrication due to rotation is shown in figure 5 and that due to squeezing – in figure 7.

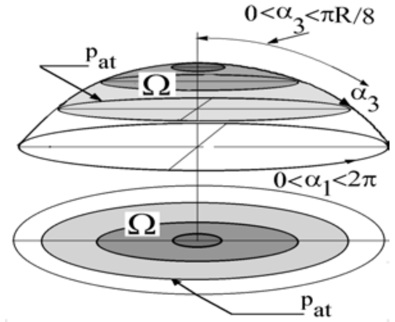


Fig. 7. Pressure-distribution region resting on surface of bone head during lubrication caused by squeezing

Equation representing spherical cartilage surface in the centre point $O_1(x - \Delta\varepsilon_1, y - \Delta\varepsilon_2, z + \Delta\varepsilon)$ can be written as:

$$(x - \Delta\varepsilon_1)^2 + (y - \Delta\varepsilon_2)^2 + (z + \Delta\varepsilon_3)^2 = (R + D + \varepsilon_{\min})^2,$$
$$D = [(\Delta\varepsilon_1)^2 + (\Delta\varepsilon_2)^2 + (\Delta\varepsilon_3)^2]^{0.5}. \tag{24}$$

Inserting dependencies (23) in equation (24) we obtain:

$$(r\cos\varphi\sin\alpha_3/R - \Delta\varepsilon_1)^2 + (r\sin\varphi\sin\alpha_3/R - \Delta\varepsilon_2)^2$$
$$+ (r\cos\alpha_3/R + \Delta\varepsilon_3)^2 = (R + D + \varepsilon_{\min})^2. \tag{25}$$

Gap height has the following form:

$$\varepsilon(\varphi, \alpha_3/R) \equiv u_2 + r - R\,. \tag{26}$$

We find function r from equation (25) and insert it in formula (26). The gap height has finally the following form:

$$\varepsilon(\varphi, \alpha_3/R) = u_2 + \Delta\varepsilon_1\cos\varphi\sin(\alpha_3/R) + \Delta\varepsilon_2\sin\varphi\sin(\alpha_3/R)$$
$$- \Delta\varepsilon_3\cos(\alpha_3/R) - R + \{[\Delta\varepsilon_1\cos\varphi\sin(\alpha_3/R) + \Delta\varepsilon_2\sin\varphi\sin(\alpha_3/R)$$
$$- \Delta\varepsilon_3\cos(\alpha_3/R)]^2 + (R + \varepsilon_{\min})(R + 2D + \varepsilon_{\min})\}^{0.5}. \tag{27}$$

The minimum of gap height for a spherical hip joint we obtain from the formula [5]:

$$\frac{\varepsilon_{\min}}{R} \equiv \sqrt[5]{2\pi}\, S_1^{0.4}\left(\frac{\omega R^2\eta}{C}\right)^{0.6}, \quad S_1 \equiv \frac{C}{ER^2}, \quad \frac{1}{E} = \frac{1}{2}\left(\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}\right), \tag{28}$$

where E_1, E_2, ν_1, ν_2 are elastic module and the Poisson ratio for bone head and cartilage, respectively, C is the load, and the quantities η, ω, R are defined previously. Dependence (1) for $\Theta \approx \omega R/\varepsilon_{\min}$ can be written in the following form:

$$\frac{\omega R^2\eta}{C} \equiv \frac{S_2}{S_1}\left(\frac{\eta_\infty}{\eta_0} + \frac{\dfrac{\eta_0-\eta_\infty}{\eta_0}}{1+S_3\dfrac{R}{\varepsilon_{\min}}}\right), \quad S_2 \equiv \frac{\omega R\eta_0}{ER}, \quad S_3 \equiv A\omega. \tag{29}$$

Combining equations (28),(29) we obtain the system of two equations determining two unknown quantities, namely dynamic viscosity η of synovial fluid and minimal value $\varepsilon_{\min}$ of gap height. In these equations, elastic deformations of cartilage are taken into account. If we assume that $R = 2.6 \times 10^{-2}$ m, $E = 2 \times 10^5$ Pa, $\omega R = 3 \times 10^{-1}$ m/s, η_∞ = 0.10 Pas, $2\pi R/C = 3 \times 10^{-4}$ m/N, $\eta_0/\eta_\infty \cong 1000$, A = 1.88 s, C = 544.26 N, then we obtain $\varepsilon_{\min}$ = 0.0000208 m = 20.88 μm and η = 0.1036 Pas. If the following quantities: A = 1.88 s, η_0 = 100.00 Pas, η_∞ = 0.10 Pas, R = 0.020 m, C = 544 N, $0.50\ \text{s}^{-1} \le \omega \le 10.00\ \text{s}^{-1}$, $2 \times 10^5\ \text{Pa} \le E \le 2 \times 10^7$ Pa are involved in computations, then we obtain minimal value of gap height in the interval of 0.29 μm ≤ $\varepsilon_{\min}$ ≤ 19.90 μm.

6. Numerical calculations of pressure distributions and carrying capacities in deformed human spherical hip joint

6.1. Formulae for carrying capacity forces

Total carrying capacity force exerted on curvilinear bone head is given by the surface integral from following formula:

$$C_{\text{tot}} \equiv \iint_{\Omega(\alpha_1,\alpha_3)} p(\alpha_1,\alpha_3)\, d\Omega(\alpha_1,\alpha_3). \tag{30}$$

Area element in the double integral has the following form:

$$d\Omega \equiv \left|\frac{\partial \mathbf{r}_0}{\partial\varphi} \times \frac{\partial \mathbf{r}_0}{\partial\vartheta}\right|_{r=R} d\varphi\, d\vartheta. \tag{31}$$

Symbol $\mathbf{r}_0$ denotes radius vector for bone head surface; $0 < \alpha_1 \equiv \varphi < 2\pi c_1$, $0 < c_1 < 1$, $\pi R/8 < \alpha_3 \equiv \vartheta < \pi R/2$ for rotation, and $0 < \alpha_3 \equiv \vartheta < \pi R/18$ for squeezing. Radius vector $\mathbf{r}_0$ we obtain from the formula**:**

$$\mathbf{r}_o = \mathbf{i}\, x + \mathbf{j}\, y + \mathbf{k}\, z\,, \tag{32}$$

where **i**, **j**, **k** are unit vectors and for spherical coordinates we have:

$$x = R\cos\varphi\sin\vartheta/R, \quad y = R\sin\varphi\sin\vartheta/R, \quad z = R\cos\vartheta/R\,. \tag{33}$$

If we insert dependence (33) in (32) and afterwards (32) in formula (31), then we obtain:

$$d\Omega = R^2\sin(\vartheta/R)d\varphi d(\vartheta/R). \tag{34}$$

6.2. The method of numerical calculations

Partial differential equations (16), (19) of the second order and elliptical kind representing hydrodynamic lubrication by rotation and hydrodynamic lubrication by squeezing are examined numerically in spherical coordinates. Numerical calculations were done using Mathcad 2000 Professional Program and the Method of Finite Differences. This method satisfies the requirement of stability of numerical solutions of pressure function in the partial differential modified Reynolds equations of the second order with variable coefficients in the form (16) and (19). We impose atmospheric pressure on the curvilinear boundaries of the region Ω resting on the surface of head of bone in human hip joint.

Dynamic viscosity of synovial fluid decreases with the increase of the shear rate. Shear rate of synovial fluid increases if angular velocity ω of head of human hip joint increases or joint gap height decreases. In the present calculations, these changes are taken into account.

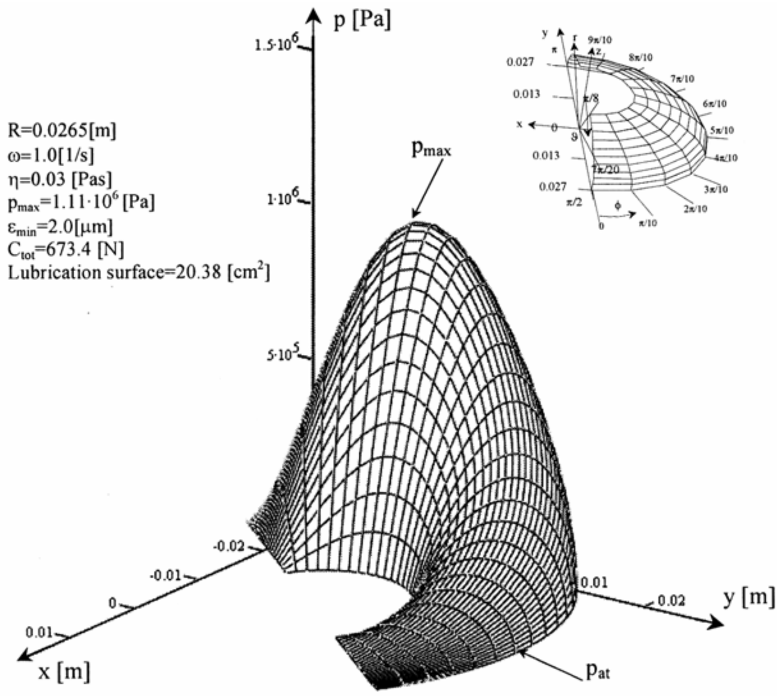
6.3. Capacity forces for hydrodynamic lubrication by rotation

If magnetic field is neglected, then modified Reynolds equation (19) for hydrodynamic lubrication caused by rotation of spherical bone head has the following form:

$$\frac{\partial}{\partial\varphi}\left(\frac{\varepsilon^3(u_2)}{\eta_0}\frac{\partial p^{(0)}}{\partial\varphi}\right) + R^2\sin\left(\frac{\vartheta}{R}\right)\frac{\partial}{\partial\vartheta}\left[\frac{\varepsilon^3(u_2)}{\eta_0}\frac{\partial p^{(0)}}{\partial\vartheta}\sin\left(\frac{\vartheta}{R}\right)\right] = 6\omega R^2\frac{\partial\varepsilon(u_2)}{\partial\varphi}\sin^2\left(\frac{\vartheta}{R}\right), \tag{35}$$

where $0.30\pi < \alpha_1 \equiv \varphi < 1.30\pi$, $\pi R/8 < \alpha_3 \equiv \vartheta < \pi R/2$.

In numerical calculations, we assume the following values for joint gap: $\Delta\varepsilon_1 = 2$ μm, $\Delta\varepsilon_2 = 2$ μm, $\Delta\varepsilon_3 = +2$ μm, radius of bone head $R = 0.026575$ m and a boundary of the region $\Omega(\alpha_1, \alpha_3)$ resting on bone head is subjected to atmospheric pressure (see figure 5). For a normal hip joint we assume in calculations the smallest gap height $\varepsilon_{\min}$ = 2.0 μm. Taking into account the angular velocity of bone head $\omega = 1\ \mathrm{s}^{-1}$ and an average value of dynamic viscosity of synovial fluid η_0 = 0.03 Pas, we obtain from equation (35) that hydrodynamic pressure $p^{(0)}$ has maximal value equal to 1.11×10^6 N/m^2 and capacity C_{tot} = 673 N. Taking into account the angular velocity of bone head $\omega = 0.1\ \mathrm{s}^{-1}$ and an average value of dynamic viscosity of synovial fluid η_0 = 0.40 Pas, we obtain from equation (35) that hydrodynamic pressure $p^{(0)}$ has maximal value equal to 1.44×10^6 N/m^2 and carrying capacity C_{tot} = 897 N (see figure 8). Lubrication surface has value $\pi R^2 \cos\pi/8 \approx 20.50$ cm^2.



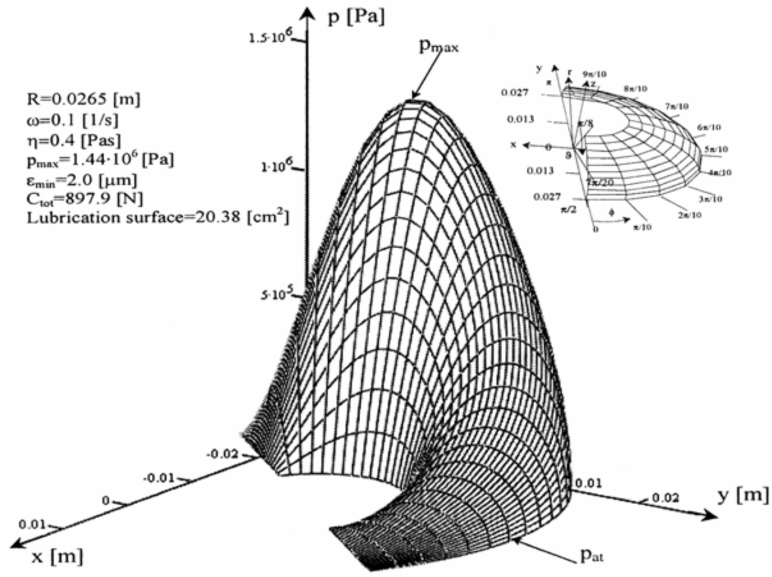


Fig. 8. Two cases of pressure distribution in normal spherical hip joint gap during hydrodynamic lubrication caused by rotation

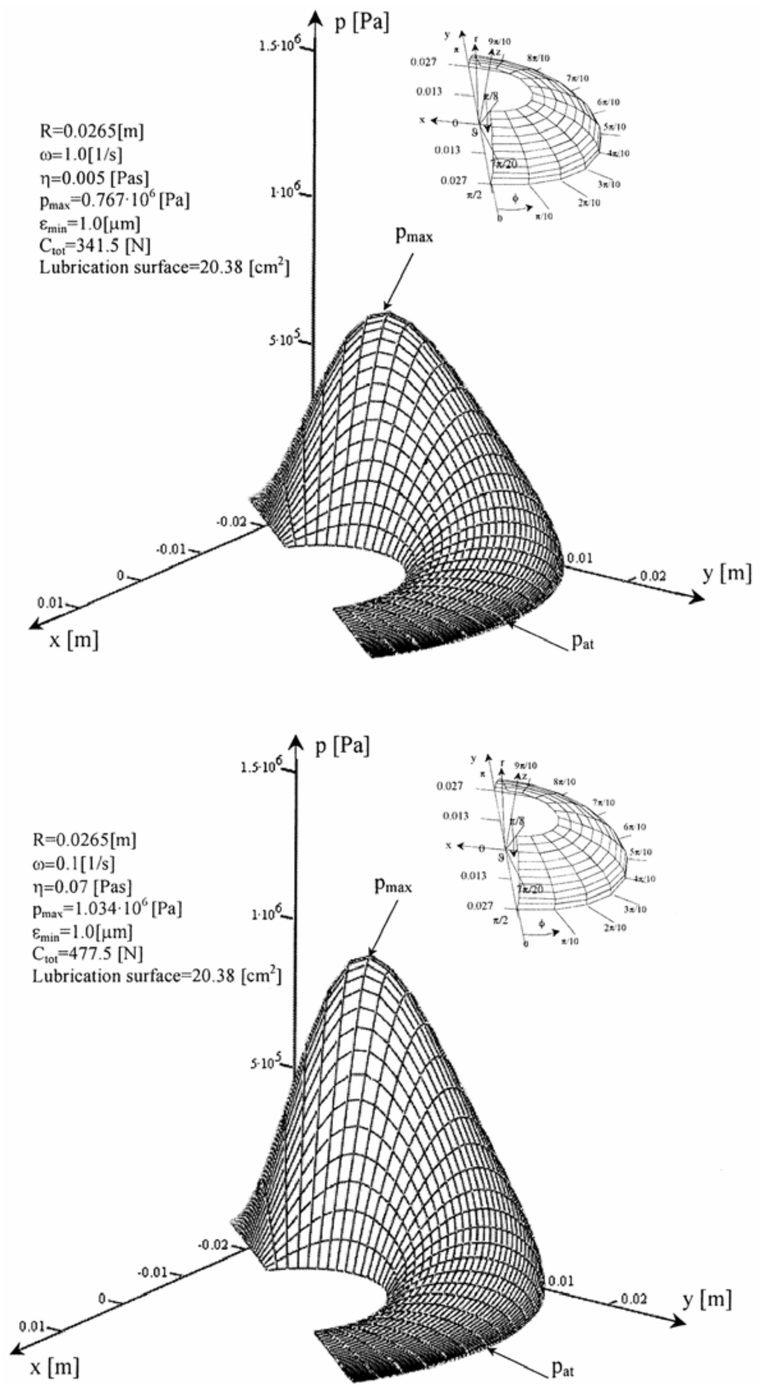


Fig. 9. Two cases of pressure distribution in pathological spherical hip joint gap during hydrodynamic lubrication caused by rotation

For a pathological hip joint we assume in calculations the smallest gap height ε_{min} = 1.0 μm. For the angular velocity of bone head $\omega = 1\ s^{-1}$ and an average value of dynamic viscosity of synovial fluid η_0 = 0.005 Pas, we obtain from equation (35) that hydrodynamic pressure $p^{(0)}$ has maximal value equal to 0.76×10^6 N/m^2 and carrying capacity C_{tot} = 341 N. Taking into account the angular velocity of bone head ω = 0.1 s^{-1} and an average value of dynamic viscosity of synovial fluid η_0 = 0.07 Pas, we obtain from equation (35) that hydrodynamic pressure $p^{(0)}$ has maximal value equal to 1.034×10^6 N/m^2 and carrying capacity C_{tot} = 477.5 N. These pressure distributions on bone head for gaps of the normal and pathological human hip joints are shown in figures 8 and 9, respectively.

For the capacities of 897 N and 673 N occurring in normal joint we obtain the following compressive stresses: σ = 897 N/20.4 cm^2 = 0.43 N/mm^2 = 0.43 MN/m^2 and σ = 673 N/20.4 cm^2 = 0.33 N/mm^2 = 0.33 MN/m^2. In pathological joint, compressive stresses are as follows: σ = 341 N/20.4 cm^2 = 0.16 N/mm^2 = 0.16 MN/m^2 and σ = 477.5 N/20.4 cm^2 = 0.23 N/mm^2 = 0.23 MN/m^2. These stresses are smaller than compressive strength of 21 MN/m^2 of human bone [1], [3]–[6].

6.4. Capacity forces for hydrodynamic lubrication by squeezing

If magnetic field is neglected, then the Reynolds equation (16) for hydrodynamic lubrication caused by squeezing between spherical bone heads has the following form:

$$\frac{1}{R\sin\frac{\vartheta}{R}}\frac{\partial}{\partial\phi}\left(\frac{\varepsilon^3}{\eta_0}\frac{\partial p}{\partial\phi}\right)+\frac{\partial}{\partial\vartheta}\left(R\sin\left(\frac{\vartheta}{R}\right)\frac{\varepsilon^3}{\eta_0}\frac{\partial p}{\partial\vartheta}\right)=-12UR\sin\frac{\vartheta}{R}, \tag{36}$$

$0 < \alpha_1 \equiv \phi \le 2\pi,\ 0 < \alpha_3 \equiv \vartheta \le R\pi/18,\ 0 < \alpha_2 \equiv r \le \varepsilon.$

In calculations, we assume the following parameters for human joint gap: $\Delta\varepsilon_1$ = –5 μm, :$\Delta\varepsilon_2$ = –5 μm, :$\Delta\varepsilon_3$ = +5 μm, the radius of bone head R = 0.026575 m and a boundary for region $\Omega(\alpha_1, \alpha_3)$:$\{0 < \alpha_1 \equiv \phi \le 2\pi,\ 0 < \alpha_3 \equiv \vartheta \le R\pi/18\}$ resting on bone head is subjected to the atmospheric pressure (see figure 7). Taking into account uniform velocity of the bone head U = 0.05 m/s, the smallest gap height ε_{min} = 15 μm and an average value of the dynamic viscosity of synovial fluid η_0 = 0.03 Pas, we obtain from equation (36) that hydrodynamic pressure $p^{(0)}$ has maximal value equal to 22.52×10^6 N/m^2 and capacity C_{tot} = 1016 N. Taking into account uniform velocity of bone head U = 0.20 ms^{-1}, the smallest gap height ε_{min} = 20 μm and an average value of the dynamic viscosity of synovial fluid η_0 = 0.01 Pas, we obtain from equation (36) that hydrodynamic pressure $p^{(0)}$ has maximal value equal to 13.37×10^6 N/m^2. The surface of lubrication is equal to $2\pi R^2[1 - \cos(\pi/18)] \approx 0.67$ cm^2, and capacity C_{tot} = 603 N (see figure 10).

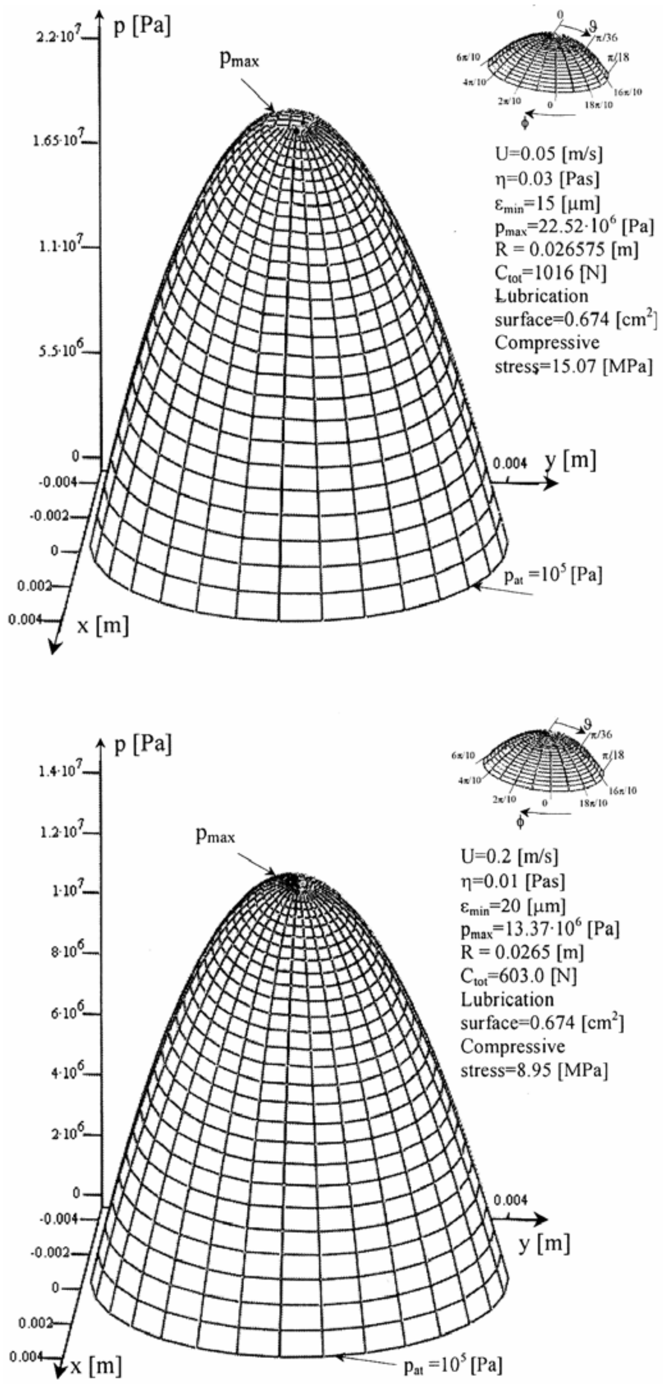


Fig. 10. Two cases of pressure distribution in normal spherical hip joint during hydrodynamic lubrication caused by squeezing

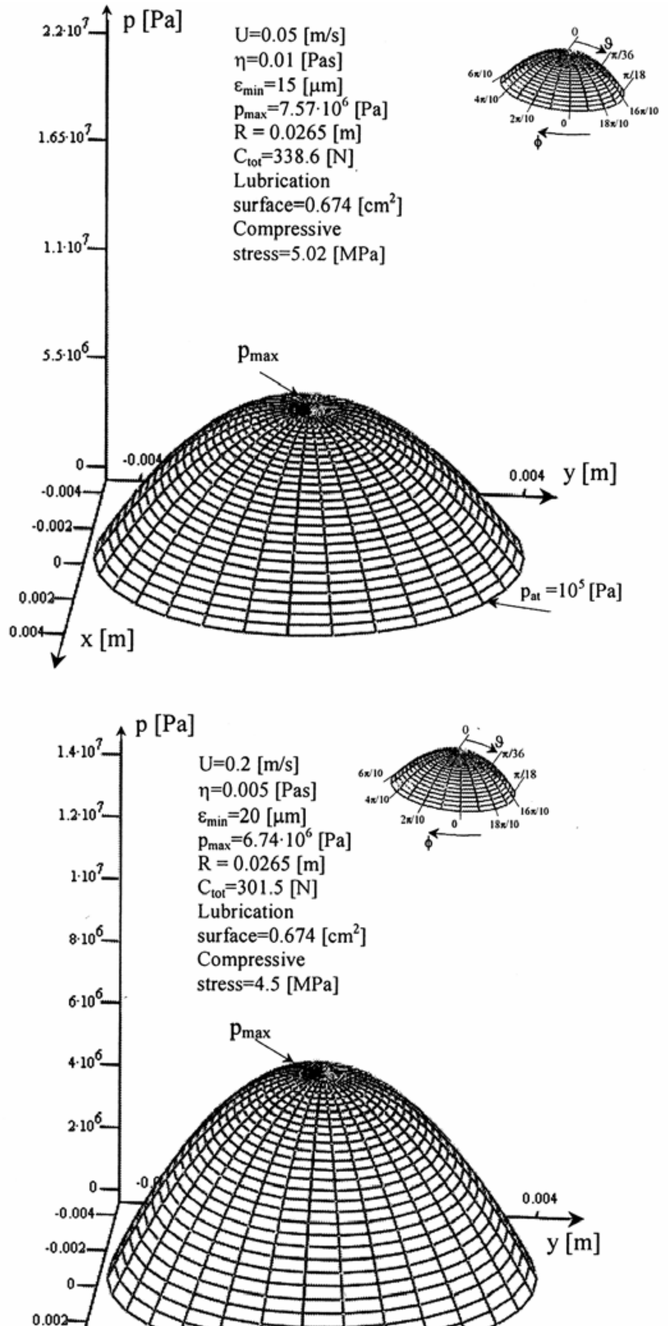


Fig. 11. Two cases of pressure distribution in pathological spherical hip joint during hydrodynamic lubrication caused by squeezing

In calculations for a pathological hip joint, we assume a uniform velocity of bone head $U = 0.05$ ms^{-1}, the smallest gap height $\varepsilon_{\min} = 15$ μm and an average value of the dynamic viscosity of synovial fluid $\eta_0 = 0.01$ Pas. Hence, from equation (36) we obtain that hydrodynamic pressure $p^{(0)}$ has maximal value of 7.57×10^6 N/m^2 and carrying capacity $C_{\text{tot}} = 338.6$ N. Taking into account uniform velocity of bone head $U =$ 0.20 s^{-1}, the smallest gap height $\varepsilon_{\min} = 20$ μm and an average value of the dynamic viscosity of synovial fluid $\eta_0 = 0.005$ Pas, we obtain from equation (36) that hydrodynamic pressure $p^{(0)}$ has maximal value equal to 6.74×10^6 N/m^2. Contact surface approaches 0.674 cm^2 and capacity is $C_{\text{tot}} = 301.5$ N (see figure 11).

For the capacities equal to 1016 N, 603 N occurring in normal joint we obtain the following compressive stresses: $\sigma = 1016$ N/0.674 cm^2 = 15.07 N/mm^2 = 15.07 MN/m^2 and
$\sigma = 603$ N/0.674 cm^2 = 8.95 N/mm^2 = 8.95 MN/m^2. In pathological joint, compressive stresses are as follows: $\sigma = 338.6$ N/0.674 cm^2 = 5.02 N/mm^2 = 5.02 MPa and $\sigma =$ 301.5 N/0.674 cm^2 = 4.5 N/mm^2 = 4.5 MPa.These stresses are smaller than bone compressive strength (21 MN/m^2) of 20- or 30-year-old human in good health. Bone compressive strength of 70-year-old human approaches 12 MN/m^2. These compressive stresses will damage the bone of 70-year-old human [1], [3]–[6].

Present method enables us to obtain solutions in the form of the Taylor series with increasing powers of small parameter A. The parameter A was obtained experimentally for synovial fluid. In particular case, for symmetrical flow we can by virtue of the present theory find analytical solutions in simple form. The percentage corrections of velocity $v_i^{(1)}$ and of pressure $p^{(1)}$ caused by the non-Newtonian properties of synovial fluid are examined numerically in the following ratio form:

$$100\frac{Ap^{(1)} + O(A^2)}{p^{(0)}} \text{ in per cent.} \tag{37}$$

For large shear rates, i.e. 100 s^{-1} $\le \Theta \le$ 1000 s^{-1}, viscosity of synovial fluid is as small as 10^{-1} Pas $\le \eta \le$ 1 Pas (see figure 6). In this case, from equation (37) we obtain small pressure changes ranging from 2% to 4%. For such small shear rates as 10^{-1} s^{-1} $\le \Theta \le$ 10 s^{-1}, the viscosity is large, i.e. 10 Pas $\le \eta \le$ 100 Pas. In this case, from equation (37) we obtain the pressure change ranging from 7% to 15%. Unsteady magnetic induction field 0.1 mT with the frequency of about 60 Hz changes the pressure distributions from 1 to 4%.

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Appendix 1

The Duhamel–Neumann relations between the components τ_{ij} of the stress tensor **S** in the elastic cartilage and the components ε_{ij} of the strain tensor have the following form [10]:

$$\tau_{ij} = 2G\varepsilon_{ij} + \left(\Lambda\varepsilon_{kk} - 3\alpha_T KT^*\right)\delta_{ij} \tag{A1}$$

for $i, j = 1, 2, 3$, where δ_{ij} is the unit Kronecker tensor component ($\delta_{ij} = 0$ for $i = j$ and $\delta_{ij} \neq 0$ for $i \neq j$). Now we consider the strain–displacement relations given in [10]:

$$\varepsilon_{ij} = \frac{1}{2}\left[\frac{h_i}{h_j}\frac{\partial}{\partial\alpha_j}\left(\frac{u_i}{h_i}\right) + \frac{h_j}{h_i}\frac{\partial}{\partial\alpha_i}\left(\frac{u_j}{h_j}\right)\right] + \delta_{ij}\sum_{k=1}^{3}\frac{u_k}{h_i h_k}\frac{\partial h_i}{\partial\alpha_k}, \tag{A2}$$

where u_i are the components of the displacement vector **u** of cartilage.

We insert the stress–strain relation, equation (A1), and strain–displacement relations, equation (A2), into the following motion and heat equations for cartilage:

$$\text{Div}\,\mathbf{S} = 0, \tag{A3}$$

$$\text{div}(\kappa^*\text{grad}T^*) = 0, \tag{A4}$$

where κ^* stands for thermal conductivity for cartilage body, and T^* is a temperature in cartilage body.

The height of elastic layer is of the order of ε_s which is about thousand times smaller than the radius of the body of cartilage curvature or other quantities occurring in the friction region. If the temperature T^* in elastic cartilage and the displacement vector **u** in cartilage are independent of time t and if the terms of the order of ε_s are neglected, then from equations (A3), (A4) we obtain the following system of partial differential equations of the second order, namely equation (14) and simplified heat transfer equation:

$$\frac{\partial}{\partial\alpha_2}\left(\kappa^*\frac{\partial T^*}{\partial\alpha_2}\right) = 0 \tag{A5}$$

for $i = 1, 2, 3$; $0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$. The height of elastic cartilage changes in the range from s_2 to s_3, i.e. $s_2 \le \alpha_2 \le s_3$.

On the internal surface of cartilage $\alpha_2 = s_2$, where an elastic layer of cartilage is in contact with the synovial fluid, the normal stresses are equal to the hydrodynamic pressure p with opposite sign, and the tangential stresses equal zero. The hydrodynamic pressure p acts vertically on the external surface of the elastic layer of cartilage and therefore the pressure is not distributed into tangential stresses on the surface. The elastic layer of cartilage is laying on a rigid bone in the place $\alpha_2 = s_3$, and therefore the contact surface of these bodies is not deformed by the pressure. The boundary conditions in the elastic cartilage have the following form:

$$\tau_{ij}(\alpha_1, \alpha_2 = s_2, \alpha_3) = -p(\alpha_1, \alpha_3)\delta_{i2}\delta_{ij}, \tag{A6}$$

$$u_i\ (\alpha_1, \alpha_2 = s_3, \alpha_3) = 0, \tag{A7}$$

where $i = 1, 2, 3$.

We insert the stress–strain relation (A1) and strain–displacement relation (A2) into equations expressing the boundary conditions (A6, A7). In these equations, we neglect the terms of the order of $\varepsilon_s / R \cong 10^{-3}$ being compared with the terms of the order of 1, where R denotes the length of the radius of the bone head. Hence we obtain equation (21).

Appendix 2

Displacement of elastic layer in gap height direction has the following form:

$$u_2(\alpha_1, \alpha_2, \alpha_3) = -\int_{\alpha_2}^{s_3} \alpha_T \cdot \frac{3K}{2G + \Lambda} \times \Delta T^*(\alpha_1, \alpha_2, \alpha_3) d\alpha_{2s} + \int_{\alpha_2}^{s_3} \frac{p(\alpha_1, \alpha_3)}{2G + \Lambda} d\alpha_{2s}, \tag{A8}$$

where: $0 \le \alpha_1 \equiv \varphi \le 2\pi c_1$, $0 < c_1 < 1$, $b_m \equiv \pi R/8 \le \alpha_3 \equiv \vartheta \le \pi R/2 \equiv b_s$, $s_2 \le \alpha_2 \le \alpha_{2s} \le s_3$, α_{2s} is an integration parameter, $\Delta T^*(\alpha_1, \alpha_2, \alpha_3)$ are the changes of the temperature in an elastic cartilage layer.

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