

A parametric model of sprinting with speed limited by rotational inertia of the legs

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The development of a simple, semi-analytical parametric model of sprinting, particularly for the 60 m indoor sprint, is described. The overall aim is to produce a tool for studying the influence of a range of physiological and external parameters on the time it takes to complete this event. The basis of the model is the hypothesis that maximum speed is limited by the fact that there is a minimum time for the sprinter's front leg to be rotationally accelerated backwards from its outstretched position to the vertical position at which the foot strikes the ground. This angular acceleration is linked via estimates of the leg's mass and moments of inertia to the linear acceleration of the athlete at the start of the race, allowing a velocity/time diagram to be constructed for the whole motion. The effect of a change to the mass of the shoe is then studied as an example of the model's usefulness.

Key words: sprinting, model, inertia

1. Introduction

Sprinting was probably one of the first forms of human competition to be formalised into a sport but it was only relatively recently, with the move towards intensive, sprinting-oriented training programmes, that much thought has been given to there being an ultimate limit to the maximum speed of a sprinter. Furthermore, it was only with the advent of the modern Olympic Games and the collation of international records that the notion was conceived that there may be different limiting speeds for different shapes of sprinters with the same overall heavily muscled, mesomorph build which typifies this event. However, it is difficult to know exactly what physical characteristics to highlight as being particularly suitable for sprinting when helping young athletes choose one event over another. The only sure way to handle such a situation is to develop a biomechanical model of a sprinter which can be used to study the event from start to finish and therefore predict the total time taken as a function of some key anatomical dimensions.

Various attempts have been made to produce models of sprinting, such as by Furusawa, Hill and Parkinson [1], Senator [2] and Keller [3]. These have hypothesised that the limiting factor on speed is the external resistance, principally wind resistance, which itself is speed dependent. In the first two cases, the resistance is taken to be directly proportional to the runner's speed, while in the third case the dependence is on any positive power of the speed. Senator goes so far as to represent his analytical model diagrammatically by a stylised sprinter dragging a trailing mass along the ground. All three of these models assume that there is a maximum force that can be developed by the sprinter. Again the model of Senator is more refined in that it uses this limiting assumption only at cruising speed and employs a different limit based on traction force during the initial acceleration.

While all these models met with some success in terms of their overall agreement with sprint data, none of them lends itself to a comparative study of different athlete body shapes. Both Keller and Senator were more concerned with being able to predict the maximum steady state rate at which a runner can develop mechanical energy and hence predict the oxygen uptake rate. Indeed, Keller [4] also applied his sprinting model to middle distance running specifically in order to verify his uptake predictions. The problem essentially is that the models are completely based on smooth, stride-averaged functions and so there is no direct mention of the body's internal dynamics involving mass and moments of inertia through such parameters as leg dimensions.

The alternative approach of starting with a very detailed biomechanical description of the foot and leg [5], and building up to a complete biomechanical model of the running process [6] includes so many poorly quantifiable anatomical and physiological parameters that it is difficult to apply.

More importantly, perhaps, the whole concept of times being effectively limited by some external resistance seems to be poorly founded. Clearly there will be external forces, such as air resistance and frictional drag, but these have proved to be difficult to measure in experiments comparing over ground running with treadmill running [7], particularly since there are fundamental differences between these two forms of running [8]. Furthermore, these external resistance forces are likely to be small judging by the long distances (anything up to 30 m) it takes for athletes to slow down from a sprint to a walking pace at the end of a race.

A more attractive hypothesis is to postulate that the sprinter's maximum speed is limited by there being a minimum time for the legs to perform a complete sprint stride movement. In particular, there will be a limit imposed by the time it takes for the leading leg to be rotationally accelerated from its most forward position to the point where it touches the ground. This limit is set by the balance between the available muscular torque and the leg's moment of inertia, both acting about the hip joint. It is therefore a limit set by internal resistance. Some anecdotal support for this notion can be gained from the experiences of sprinters who take part in downhill races for the first time and find that the extra motive force provided by gravity soon results in

overbalancing and falling. The component of their weight along the slope more than overcomes any resistance and should lead to a higher maximum speed than on the flat, but there is not enough time in the first portion of the stride cycle for the leading leg to accelerate to the higher speed. Quite simply, they are trying to run faster than their legs can carry them.

The purpose of the study described here was therefore to develop a new simple model of sprinting according to the following criteria.

- It must not rely on indirect data, such as the values of the various resistances or the sprinter's developed mechanical power, as these are problematic [9].
- It must be in overall agreement with race times.
- It should be parametric so that the effect of changes in e.g. leg shape or shoe mass may easily be identified, making it a useful analytical tool.
- It should be largely analytical so that the influence of individual parameters can be more easily perceived.
- It should build on successful aspects of earlier models.
- It should be based on the maximum speed being limited by the moment of inertia of the leg.

2. Model construction

A sprint race is considered in the model to comprise three sections: the start, the rise from a crouch to erect running and the uniform run. The rise is taken to be a transition phase and so it will be dealt with last.

2.1. The start

Although the model of Senator [2] is not deemed to be suitable for the parametric study here, as discussed earlier, it showed clearly that the start of an indoor sprint seems to be dominated by the coefficient of friction, or traction, between shoes and track. In other words, the coefficient of friction achieves a limiting value μ and the shoe in contact with the track is on the point of slipping. This assumption will be adopted here initially and so the force F available to accelerate the sprinter away from the start is given by:

$$F = \mu Mg, \quad (1)$$

where M is the sprinter's mass and g is the acceleration due to gravity. Note that the concept of this force is based on Senator's idea of a stride-averaged value but here, once the rear foot has lifted off, it really is the tractive force exerted by one foot.

Using Newton's second law, the maximum acceleration a_{\max} becomes:

$$a_{\max} = \mu g. \quad (2)$$

2.2. Uniform running

The maximum speed of uniform running is hypothesised to be limited by the ability of the leg muscles to accelerate the leading leg from the point where it is straight, outstretched in front and at rest, relative to the sprinter, to the point where the foot strikes the ground. Analysis of high-speed video data of uniform running indicates that the point of impact or contact with the track is approximately vertically below the sprinter's centre of gravity. The leg moves through an angle of θ , as shown in figure 1, and at the point of impact the foot must have reached the same linear speed v_{\max} as the sprinter in order to maintain uniform motion.

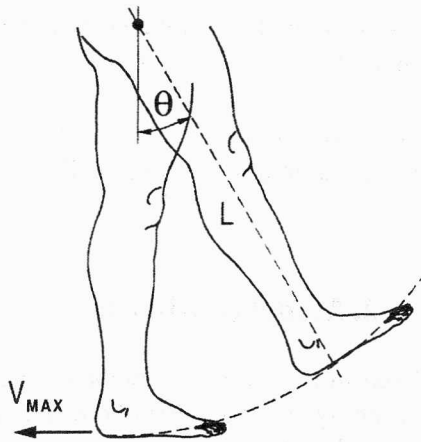


Fig. 1. Nomenclature for the leading leg during uniform running

If leg length from the pivot point of the hip to the ball of the foot is L then the angular velocity of the foot at the point of impact is given by v_{\max}/L .

Assuming that the angular acceleration is constant and giving it a value of α_{\max} to denote that the sprinter is running at maximum speed, then from the equation of uniform angular motion for a body rotating from rest:

$$\left(\frac{v_{\max}}{L}\right)^2 = 2\alpha_{\max}\theta. \quad (3)$$

In order to evaluate v_{\max} there are two geometric terms, L and θ , which must be evaluated. The angular acceleration also needs to be related to the individual athlete. This is done by supposing that it results from the same maximum transferable force as in equation (1), acting tangential to the foot and therefore at a lever arm distance of L from the hip. Consequently:

$$FL = I_0\alpha_{\max}. \quad (4)$$

I_0 is the moment of inertia of the extended leg about the hip.

Finally the maximum velocity of the sprinter, equal to the maximum linear velocity of the foot, is:

$$v_{\max} = L \left(\frac{2\mu MgL\theta}{I_0} \right)^{1/2}. \quad (5)$$

2.3. The rise

During the rise, the sprinter must produce two types of force of interest here. The first is the external tractive force on the ground required to produce a linear acceleration a of the whole body; the second is the internal force to produce sufficient torque to give the leading outstretched leg an acceleration of α so that it keeps up with the accelerating body. The sum is taken to be equal to the maximum transferable force found in equation (1). Therefore:

$$F = \mu Mg = Ma + \frac{I(\beta)\alpha}{h(\beta)}. \quad (6)$$

$I(\beta)$ is the moment of inertia of the bent leg about the hip and $h(\beta)$ is the height of the hip pivot point above the ground, as indicated in figure 2. Both of these terms vary with β , the angle through which the knee has flexed away from the fully extended position. If the knee is taken to be approximately half way along the leg (as shown in the next section) then:

$$h(\beta) = L \cos \frac{\beta}{2}. \quad (7)$$

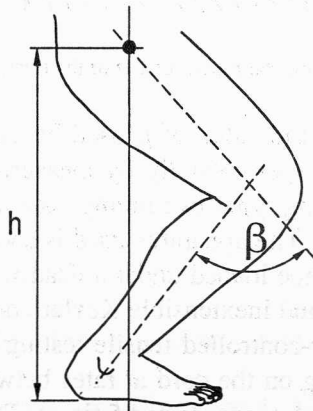


Fig. 2. Nomenclature for the leading leg during the rise

Remembering that $a = h(\beta)\alpha$ we have finally:

$$a = \frac{\mu g M}{M + \frac{I(\beta)}{\left(L \cos \frac{\beta}{2}\right)^2}}. \quad (8)$$

3. Solution

The formulation of the three parts of the model has now been described, but it is by no means straightforward to produce a velocity/time diagram from them yet. First, some assumptions and measurements have to be made to put numerical values to all the parameters. Secondly, since equation (8) produces a value of acceleration, there has to be an integration to give a velocity.

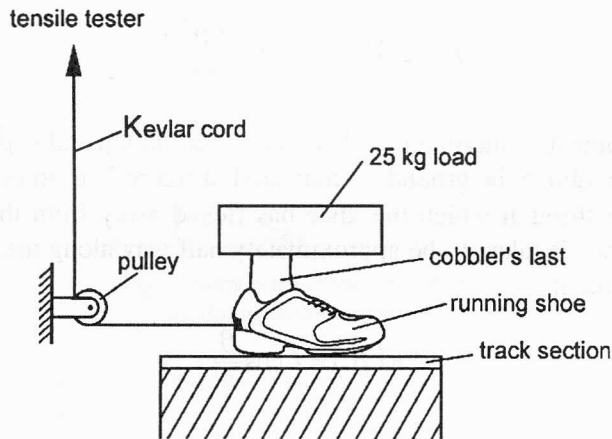


Fig. 3. Experimental measurement of the traction coefficient

Rather than simply adopt the value of μ used by Senator [2], i.e. 0.9, it was decided to confirm this value experimentally by measuring the limiting coefficient of friction between several leading types of running shoe and a plank of sealed wood as used to construct sports halls. The apparatus used is shown schematically in figure 3. It consisted essentially of a shoe loaded under a dead weight onto a track section and pulled backwards via a light and inextensible Kevlar cord passing under a small, low-friction pulley to a computer-controlled tensile testing machine. Traction force was applied to the shoe by pulling on the cord at rates between 2 and 20 mm/s and each combination of load rate and shoe was tested 5 times. The force applied by the tensile tester was recorded and presented as a graphical output from which it was possible to estimate the maximum force at the point of incipient slip. Dividing by the applied

weight gave a value of traction coefficient. It was found that any difference between the values for the various brands and load rates was smaller than the spread for repeated tests, which itself was small, and so it was sensible to produce a single value of the coefficient from all the tests. This was 0.90 ± 0.02 .

The angles θ and β were found by studying a high-speed video of an international outdoor 100 m sprint race. The reason for using a 100 m event was to ensure that maximum speed had been reached by the last part of the race; the athlete assisting with the study had expressed some doubt that indoor sprinters achieved full speed over the shorter 60 m track. For the rise, the angle β was found to vary from 60° at the beginning to zero at full speed, when the angle θ was found to be 30° . There was little variation from one competitor to another.

The remaining data were specific to one person, a 21-year old male student, who competed in sprint events at inter-University level, with mass M of 77.73 kg and a leg length L of 1.00 m. The final data required were the values of the moment of inertia of one of the athlete's legs about the hip joint for a range of knee flexion angles. In principle it would be possible to measure these using the kind of experimental method employed successfully by various researchers and reviewed by Erdmann [10]. However, this would not produce the parametric model which is required. Therefore the moments of inertia for the leg about the hip for varying degrees of knee flexion were calculated by approximating the athlete's leg to a series of truncated cones, as shown in figure 4, with pin joints at the hip and knee and a point mass representing the foot.

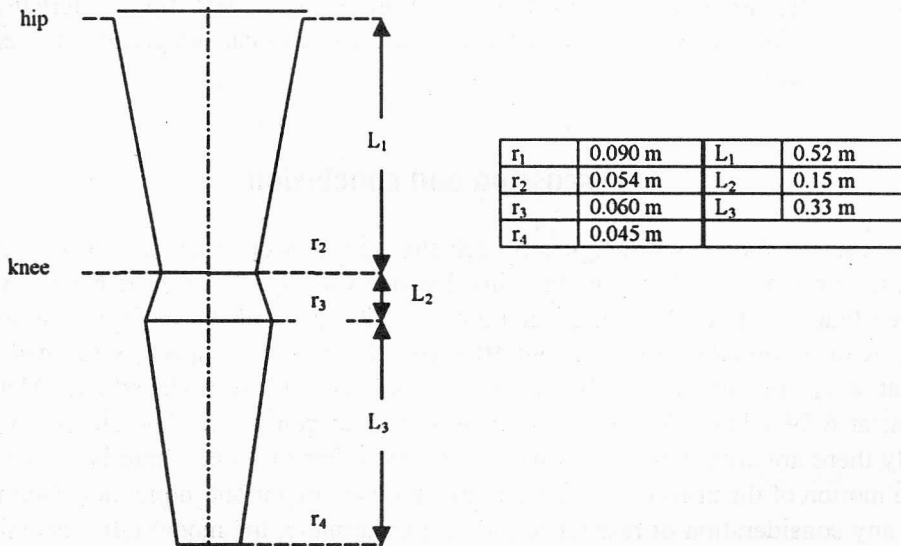


Fig. 4. Dimensions of the stylised leg

A density of 1020 kg/m^3 was used throughout, based on an Archimedean displacement experiment with a porcine leg. The foot mass was based on a displaced volume measurement, giving a value of 1.30 kg. The radii of the cones were calcu-

lated from measured circumferences of the athlete's legs. The moments of inertia of the individual parts of the leg were calculated and then combined using the parallel axis theorem so that the moment of inertia of the whole leg about the hip could be calculated for any value of knee flexion.

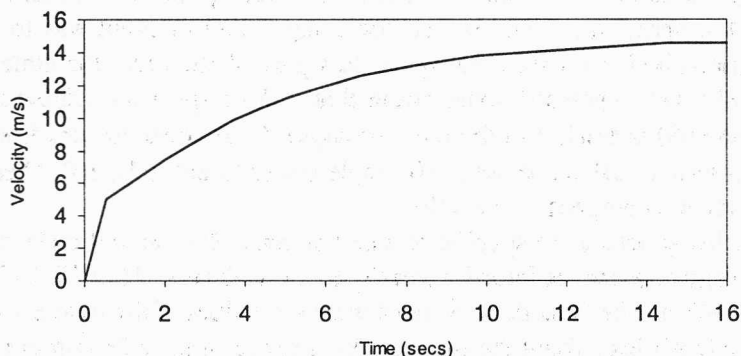


Fig. 5. Velocity-time diagram for the chosen sprinter

Once the data had been gathered it was possible to start the analysis of the sprinting itself. The velocity-time diagram in figure 5 was constructed by taking the initial slope from the *start* section and v_{\max} from the *uniform running* section. In between, the acceleration a was calculated on a spreadsheet for values of β between zero and 60° in 5° intervals. By assuming uniform motion during each of these small intervals, the time for each interval was calculated and hence a numerical integration was effectively carried out.

4. Discussion and conclusion

By limiting the curve in figure 5 such that the area underneath it was equal to 60 m, the time predicted to complete this distance was calculated to be 6.80 s. It will be seen that the athlete is still accelerating at this time and it is only in the 100 m event, with its running time of about 10 s, that the maximum speed is reached. The current world record for this 60 m event indoors is held by a US athlete, Maurice Green, at 6.39 s [11]. Agreement is therefore good considering the simplicity, but clearly there are areas which are open to criticism. For example, there is no mention of the motion of the arms or trunk even though this is important in practice. Neither is there any consideration of resistance forces. Furthermore, the model relies heavily on generic measured values for β and θ , i.e. there would be no attempt to measure these for an individual sprinter. Most particularly, perhaps, it is notable that the whole model is dependent on the traction coefficient, not just the acceleration phase. This arises from the fact that the maximum force that can be applied by the foot during the start is set equal to the maximum equivalent force at the foot that can be exerted by

the leading leg to produce its rotational acceleration during uniform running. A maximum speed that depends on traction coefficient when there are no external resistances is contrary to expectation and it may be that Senator's conclusion about μ setting the limit during acceleration is wrong. It may be simply coincidence that when the foot is exerting its maximum traction force on the ground it is also on the point of slipping. This would go some way to explaining the fact that there is little difference between accelerations and speeds recorded on indoor wooden tracks and those recorded outdoors with more traditional tracks and spiked shoes. Support for the notion of this coincidence is provided by noting that the model does not include any reference to starting blocks and so the initial acceleration is determined at present by the traction coefficient. If the link to traction coefficient is just coincidence and the limit is really set by the strength of the legs then the model would be unaffected by explicit treatment of the starting blocks.

In spite of the drawbacks considered above, the study meets the stated aim of producing a simple, parametric model of sprinting which is in overall agreement with event times and could be used to demonstrate the effect of small changes to the shape of the legs or to the mass of the shoe. As an example of this usefulness, the effect of the running shoe mass will be analysed.

First a typical running shoe was weighed and found to have a mass of 0.350 kg. This mass was then added to that of the foot in the model described above, giving a slightly longer finish time which was compared to that for a similar situation but with a hypothetical shoe which had a mass of 0.280 kg, i.e. 20% less. The reduction in time caused by this practicable reduction in mass of the shoe was 0.054 s. This might seem very small but in the context of this event it corresponds to a clear lead of approximately 0.4 m over an identical competitor with the heavier shoe.

The model therefore appears to be useful, in spite of its simplicity and shortcomings, and it is planned to extend it to study legs delivering the same overall traction force but with differing distributions of muscle. The main comparison of interest would be between two legs of the same mass but with one having a more heavily muscled thigh and the other having a more heavily muscled lower leg.

References

- [1] FURUSAWA K., HILL, A.V., PARKINSON J.L., *The Dynamics of Sprint Running*, Royal Society Proceedings B, 1927, Vol. 102, 29–41.
- [2] SENATOR M., *Extending the Theory of Dash Running*, Journal of Biomechanical Engineering – Transactions of ASME, 1982, Vol. 104, 3, 209–213.
- [3] KELLER J.B., *Optimal velocity in a race*, American Mathematical Monthly, 1974, 81, 5, 474–480.
- [4] KELLER J.B., *A theory of competitive running*, Physics Today, 1973, 26, 42–47.
- [5] De WIT B., De CLERCQ D., AERTS P., *Biomechanical analysis of the stance phase during barefoot and shod running*, Journal of Biomechanics, 2000, 33, 269–278.
- [6] MANN R., *Biomechanics of Running: Symposium on the foot and leg in running sport*, C.V. Mosby, St Louis, MO., 1982.

- [7] FRISHBERG B.A., *An analysis of overground and treadmill sprinting*, Med. Sci. Sports Exerc., 1983, 15 (6), 478–485.
- [8] NIGG B.M., *A kinematic comparison of over ground and treadmill running*, Med. Sci. Sports Exerc., 1995, 27 (1), 98–105.
- [9] JOHNSON D., BAHAMONDE R., *Effects of gender and the Lewis formula in mechanical power estimates*, Proc. XIII Int. Symp. Biomechanics in Sports (Ed. T. Bauer), Thunder Bay, Ontario, Canada, 1995, 194–197.
- [10] ERDMANN W.S., *Geometry and inertia of the human body – a review of research*, Acta of Bioengineering and Biomechanics, 1999, 1, 1, 23–35.
- [11] 3rd Feb 1998, Source – British Athletics Federation.