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Original paper

Friction force and pressure calculations for time-dependent impulsive intelligent lubrication of human hip joint

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The core of the present problem was to indicate the values of the optimum synovial fluid velocity and friction forces appearing near the cartilage cells of cooperating surfaces, as well as to find the ways of controlling the friction forces between particles of the liquid and the cells of the biobearing cooperating bodies in the thin boundary layer. In the research, we used a numerical method. Investigations of the physical and strength features are expected to be performed for various kinds of biobearing superficial layer and for damage to biobearings. In order to begin treating the surface structure of a superficial layer on the cooperating biobearing surfaces it is absolutely necessary to build a proper model of a liquid flow in the thin layer and to obtain the proper values of friction forces.

Key words: *intelbio, intelligent joint systems, time-dependent friction, Matlab calculations*

1. Introduction

The paper demonstrates the following three new groups of research: 1. The method of solving unsteady hydrodynamic lubrication for human hip joints between bone head and acetabulum. 2. Analytical determination of carrying capacity values and friction forces for unsteady flow of synovial liquid in human hip joint gap for various joint geometry. 3. New numerical methods of non-Newtonian unsteady lubrication in human hip joint using intelligent synovial fluid. The second group contains the author's foregoing analytical methods. It seems that the pressure distributions and friction forces perfectly satisfy the equation of motion and all given boundary conditions. The above problems are the continuation of scientific research carried out by the author [10]–[15], especially the calculation method that a in spherical coordinate system allows one to avoid the non-convergent Bessel functions.

We assume that spherical bone head in human hip joint moves at least in two directions: circumferential and meridional. Basic equations describing synovial fluid flow in human hip joint are solved in the analytical and numerical ways. The numerical calculations are performed in Mathcad 12 Professional Program, with taking into account the finite differences method. This method satisfies the stability conditions of numerical solutions of partial differential equations and gives real values of fluid velocity components and friction forces occurring in human hip joints.

2. Basic equations

Many lubrication theories for diarthrodial hip joints have been proposed, but a theoretical model of joint lubrication that can be used under impulsive and periodic conditions of joint unsteady motion has not fully been formulated as yet [1]–[5]. A comparison of periodic viscoelastic lubrication with impulsive one of human joint was not considered in the papers published [2], [3], [4], [7]–[14]. In this paper, lubrication

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occurs near two co-operating hip joint surfaces suddenly set in motion after an impulse. Synovial fluid has non-Newtonian properties according to Dowson's investigations [1]. The Rivlin–Ericksen constitutive equations have been used for the description of such a fluid. Bone head is often ellipsoidal in shape, but the difference between its semi-minor and semi-major axes cannot be greater than the minimum value of gap height to make the rotary motion possible [1]. Thus, for normal hip joint we can assume a spherical shape of bone head. Spherical bone head can be put into rotary motion in one or two different directions (figure 1).

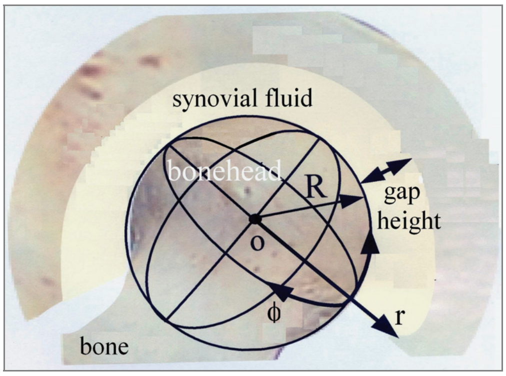


Fig. 1. Spherical bone head in human hip joint:
φ – coordinate in the circumferential direction,
r – coordinate in the gap height direction,
ϑ – coordinate in the meridional direction

For synovial fluid flow in joint gap, three-dimensional components v_φ, v_r, v_ϑ of velocity in three directions φ, r, ϑ are considered. The symbols v_φ, v_r, v_ϑ denote synovial fluid velocity components in the circumferential, gap height and meridional directions of bone head, respectively. The pressure p depends on the variable: φ, ϑ and the time t variable. The gap height ε may be a function of three variables: φ, ϑ and t. The basic equations describing synovial fluid flow in the gap of a human joint during impulsive and unsteady motion of human limbs are solved in an analytical and numerical ways. The numerical calculations are performed in Mathcad 12 Professional Program with taking into account the finite differences method. This method satisfies the stability conditions of numerical solutions of partial differential equations and gives real values of pressure and capacity forces occurring in human hip joints. The problem of impulsive, unsteady lubrication of human hip joint will be solved for the human joint surfaces between bone head and acetabulum by means of the equations of conservation of momentum and the continuity equation. These equations and the second-order approximation of a general constitutive equation given by Rivlin–Ericksen can be written in the following form [7], [11]:

$$\mathbf{S} = -p\mathbf{I} + \eta\mathbf{A}_1 + \alpha\mathbf{A}_1^2 + \beta\mathbf{A}_2, \tag{1}$$

where:

$\mathbf{S}$ – the stress tensor,

p – the pressure,

$\mathbf{I}$ – the unit tensor,

$\mathbf{A}_1$ and $\mathbf{A}_2$ – the first two Rivlin–Ericksen tensors,

η, α, β – three material constants ($\eta = \eta_0\eta$ denotes dynamic viscosity in Pas, and α, β are pseudo-viscosity coefficients in Pas^2),

ρ – the synovial fluid density in $\mathrm{kg/m}^3$.

The tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are given by symmetric matrices defined by [7], [11]:

$$\mathbf{A}_1 \equiv \mathbf{L} + \mathbf{L}^T, \quad \mathbf{A}_2 \equiv \text{grad}\,\mathbf{a} + (\text{grad}\,\mathbf{a})^T + 2\mathbf{L}^T\mathbf{L},$$

$$\mathbf{a} \equiv \mathbf{L}\,\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}, \tag{2}$$

where:

$\mathbf{L}$ – the tensor of fluid velocity gradient vector in s^{-1},

$\mathbf{L}^T$ – the tensor for transposing the matrix of gradient vector of a synovial fluid in s^{-1},

$\mathbf{v}$ – the velocity vector in m/s,

t – the time in s,

$\mathbf{a}$ – the acceleration vector in $\mathrm{m/s}^2$.

Symbol grad($\mathbf{a}$) denotes the tensor of rank two. The characteristic dimensional time t_0 is very short during the motion of human limbs after injury. We assume a rotational motion of human bone head at the peripheral velocity $U = \omega R$, where ω denotes the angular velocity of bone head, an unsymmetrical unsteady flow of synovial fluid in the gap, viscoelastic and unsteady properties of synovial fluid, a constant value of the synovial fluid density ρ and viscosity η, characteristic value of the gap height ε_0 of hip joint, no slip at the cartilage surfaces, and the radius R of bone head. To derive the governing equations we introduce the relative radial clearance $\psi \equiv \varepsilon_0/R$. We neglect the terms multiplied by the relative radial clearance because they are about thousand times smaller than the remaining terms. Thus, under taking into account the above-mentioned assumptions, the system of equations of motion in spherical coordinates φ, r, ϑ has the following form [11]:

$$\frac{\partial v_\varphi}{\partial t} = -\frac{1}{\rho R \sin\vartheta_1}\frac{\partial p}{\partial \varphi} + \frac{\eta_0}{\rho}\frac{\partial}{\partial r}\left(\frac{\partial v_\varphi}{\partial r}\right) + \frac{\beta}{\rho}\frac{\partial^3 v_\varphi}{\partial t \partial r^2}, \tag{3}$$

$$0 = \frac{\partial p}{\partial r}, \tag{4}$$

$$\frac{\partial v_\vartheta}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial \vartheta} + \frac{\eta_0}{\rho}\frac{\partial}{\partial r}\left(\frac{\partial v_\vartheta}{\partial r}\right) + \frac{\beta}{\rho}\frac{\partial^3 v_\vartheta}{\partial t \partial r^2}, \tag{5}$$

$$\frac{\partial v_\varphi}{\partial \varphi} + R\sin(\vartheta_1)\frac{\partial v_r}{\partial r} + \frac{\partial}{\partial \vartheta}[R v_\vartheta \sin(\vartheta_1)] = 0. \tag{6}$$

According to DOWSON [1], the lubrication and pressure distribution region spreads in the circumferential direction from the angle $\varphi = 0$ to the half perimeter of spherical bone, i.e. $\varphi = \pi$. In the meridional direction, the pressure origin region is located at the angle $\vartheta_1 = \pi/8$, (i.e. about 22 grades distant from the upper pole of spherical bone head) and contains the remaining part of the upper hemisphere up to the angle $\vartheta_1 = \pi/2$. Hence, the lubrication region is defined as follows: $0 \le \varphi \le 2\pi\theta_1$, $0 < \theta_1 < 1$, $\pi R/8 \le \vartheta \le \pi R/2$, $0 \le r \le \varepsilon$, $\vartheta_1 = \vartheta/R$, ε – the gap height.

The terms multiplied by the pseudo-viscosity coefficient β on the right-hand sides of equations (3) and (5) denote the influence of the time-variable viscoelastic properties of synovial fluid on the hip joint lubrication. The terms on the left-hand sides of the equations describe the influence of accelerations occurring in the motion on the lubrication parameters.

In both classes of motions, i.e. impulsive and periodic ones, it is not possible to obtain similar solutions, hence a series expansion with respect to a non-similarity parameter will be given.

3. Load carrying capacities in human spherical hip joints

The carrying capacities in spherical bearing are calculated from the following formula [11]:

$$C_{\text{tot}}^{(\text{sph})} = \sqrt{\left[\int_{R\pi/8}^{R\pi/2}\left(\int_0^{\varphi_k} p(\varphi,\vartheta)R(\sin\varphi)\left(\sin\frac{\vartheta}{R}\right)d\varphi\right)d\vartheta\right]^2 + \left[\int_{R\pi/8}^{R\pi/2}\left(\int_0^{\varphi_k} p(\varphi,\vartheta)R(\cos\varphi)\left(\sin\frac{\vartheta}{R}\right)d\varphi\right)d\vartheta\right]^2}, \tag{7}$$

where the symbol φ_k denotes the end coordinate of the film in circumferential direction, and $0 \le \varphi < 2\pi\theta_1$, $0 \le \theta_1 < 1$, $R\pi/8 \le \vartheta \le R\pi/2$, $\vartheta = R\vartheta_1$.

4. Gap height

The time-dependent gap height with perturbations has the following form [8], [11]:

$$\varepsilon(\phi,\vartheta,t) = \varepsilon_0\varepsilon_1(\phi,\vartheta_1,t_1) = \varepsilon^{(0)}[1 + s_1\exp(-t_0 t_1\omega_0)],$$

$$\varepsilon^{(0)} \equiv \varepsilon_\Delta - R + [(\varepsilon_\Delta)^2 + (R+\varepsilon_{\min})(R+2D+\varepsilon_{\min})]^{0,5}, \tag{8}$$

$$\varepsilon_\Delta \equiv \Delta\varepsilon_x\cos\phi\sin\vartheta_1 + \Delta\varepsilon_y\sin\phi\sin\vartheta_1 - \Delta\varepsilon_z\cos\vartheta_1.$$

In the time-dependent gap height, we transform the dependencies from the rectangular (x, y, z) to the spherical (φ, r, ϑ), co-ordinates for $\vartheta_1 = \vartheta/R$. We take into account the centre of spherical bone head O(0, 0, 0) and the centre of spherical acetabulum $O_1(x - \Delta\varepsilon_x, y - \Delta\varepsilon_y, z + \Delta\varepsilon_z)$.

The concentrated force [18] acts on the spherical surface of hyper-elastic acetabulum and generates the cartilage and gap height deformations $s(\varphi, \vartheta)$ in the radial direction. The longer the time that passes up to the impulse, the smaller the deformations multiplied by s_1 according to the exponential function. The coefficient s_1 represents the dimensionless changes of gap height caused by the impulsive load during the motion. The gap height increases, if $s_1 > 0$, and decreases, if $s_1 < 0$. The greater the concentrated force of impulse, the greater the absolute value of the coefficient s_1. The symbol ω_0 denotes an angular velocity or frequency in s^{-1} and describes the changes of time-dependent perturbations in synovial fluid impulsive flow in joint gap in its height direction.

If t_1 tends to infinity, then the gap height (8) tends to the time-independent gap height for a stationary flow. We assume the centre of spherical bone head at the point O(0, 0, 0) and the centre of spherical acetabulum at the point $O_1(x - \Delta\varepsilon_x, y - \Delta\varepsilon_y, z + \Delta\varepsilon_z)$. The eccentricity assumes the value D. The lubrication region for impulsive motion and the time-dependent gap-height changes under impulsive motion is denoted by Ω: $0 \le \varphi \le \pi$, $\pi R/8 \le \vartheta \le \pi R/2$. It is a section of the bowl of the sphere. The value of the pressure p distributed on the boundary of the region is equal to the value of the atmospheric pressure p_{at}.

5. Friction force mechanisms in human hip joint

The values of the friction forces depend on the ability of liquid to penetrate a superficial layer of cartilage in human hip joint or a superficial layer of co-operating bodies limited by the bone head and acetabulum [10]. The penetration ability is determined by the quotient of the penetration coefficient c_k (m^2) and the liquid dynamic viscosity η. Hence (m^4/Ns) is the unit of penetration ability. It seems that if the load produced by the pressure on the superficial layer increases, then the penetration ability of the body inside the layer decreases [16]. Hence, the height of boundary liquid layer, pressure and liquid penetration ability are mutually connected.

Changes of Young's modulus of the superficial layer of cooperating bodies indirectly influence the generation of friction forces which depend on the displacements of superficial layer of cooperating bodies and on the indent peak load.

We can observe that the changes of Young's modulus influence both strength variations and deformation of cooperating bodies which also changes the height of boundary liquid layer [16]. The changes of the height of liquid boundary layer cause the changes of flow velocities, i.e. the changes of velocity deformations, which implies apparent viscosity changes of non-Newtonian liquids. Changeable liquid dynamic viscosity is caused by the susceptibility of joint cartilage to liquid penetration [16]. Such dependences are shown in figure 2.

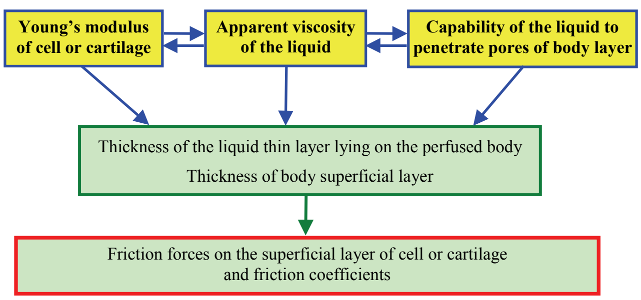


Fig. 2. Predicted influence of Young's modulus of cell or cartilage body, viscosity of the liquid and penetration susceptibility of the cartilage on the friction forces

The friction forces in the gap of human joints are shown in figures 3 and 4. They present various reasons for the generation of friction forces which depend not only on liquid penetration and liquid flow, but also on the properties of cooperating bodies, particularly those of a superficial layer of cartilage.

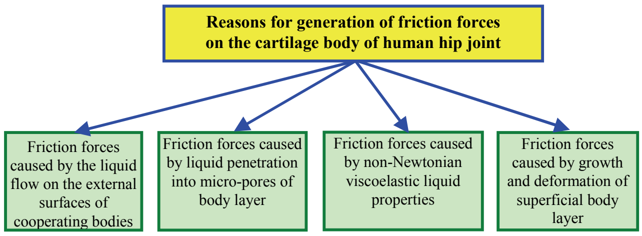


Fig. 3. Reasons for friction force generation on lubricated surfaces of human hip joint

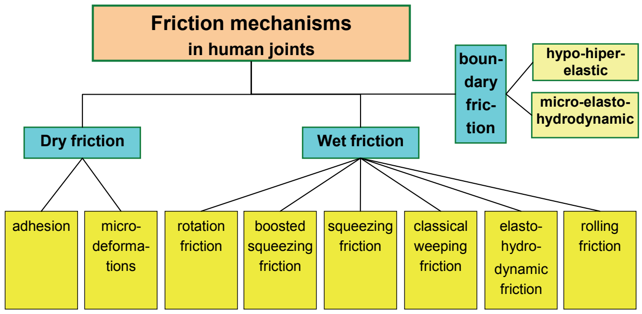


Fig. 4. Mechanisms of dry, wet and boundary friction on lubricated surfaces of human hip cartilages

Friction forces in a macroscale occur on the external surface of joint cartilage and between collagen fibers in human joints.

The thickness of joint cartilage ranges from 3 to 4 mm. Friction coefficient of a healthy human cartilage approaches 0.01 and that of pathological cartilage is 0.09 [5], [16]. Friction forces in a nanoscale are tangent to the external surfaces of glycoprotein fibers in human joints. The magnitude of such friction forces is of the order of some or over a dozen nano-Newtons. The magnitude of friction forces and their direction in superficial layer have a dominant influence on the lubrication process in human joints.

The diameters of glycoprotein fibres in superficial layer of the cartilage in human joint fibres assume the value of some nanometers.

The coefficient of friction between surfaces of fibre material and lubricant is very small, often as small as 0.0000015 [5].

The components of the friction forces are as follows [16]:

$$F_{R\varphi}(t)=\iint_{\Omega}\frac{\partial p}{\partial\varphi}\left[\varepsilon(\varphi,\vartheta)-\frac{\int_0^{\varepsilon(\varphi,\vartheta)}\frac{rdr}{\eta(\varphi,r,\vartheta)}}{\int_0^{\varepsilon(\varphi,\vartheta)}\frac{dr}{\eta(\varphi,r,\vartheta)}}\right]d\varphi d\vartheta$$

$$-\iint_{\Omega}\left[\frac{\omega R^2\left(\sin^2\frac{\vartheta}{R}\right)}{\int_0^{\varepsilon(\varphi,\vartheta)}\frac{dr}{\eta(\varphi,r,\vartheta)}}\right]d\varphi d\vartheta, \tag{9}$$

$$F_{R\vartheta}(t)=$$

$$R\iint_{\Omega}\frac{\partial p}{\partial\vartheta}\left[\varepsilon(\varphi,\vartheta)-\frac{\int_0^{\varepsilon(\varphi,\vartheta)}\frac{rdr}{\eta(\varphi,r,\vartheta)}}{\int_0^{\varepsilon(\varphi,\vartheta)}\frac{dr}{\eta(\varphi,r,\vartheta)}}\right]\sin\vartheta_1 d\varphi d\vartheta \tag{10}$$

for $\eta=\eta(\varphi, r, \vartheta)$, $0\le r\le\varepsilon$, $0\le\varphi<2\pi\theta_1$, $0\le\theta_1<1$, $R\pi/8\le\vartheta\le R\pi/2$, $\vartheta=R\vartheta_1$, $\Omega(\varphi,\vartheta)$ – lubrication surface.

If liquid dynamic viscosity is constant in gap-height direction, i.e. $\eta(\varphi,\vartheta)$, then the friction force components (9), (10) tend to the following forms [16]:

$$F_{R\varphi}(t)=\frac{1}{2}\iint_{\Omega}\varepsilon(\varphi,\vartheta)\frac{\partial p}{\partial\varphi}d\varphi d\vartheta$$

$$-\omega R^2\iint_{\Omega}\frac{\eta(\varphi,\vartheta)}{\varepsilon(\varphi,\vartheta)}\sin^2\left(\frac{\vartheta}{R}\right)d\varphi d\vartheta, \tag{11}$$

$$F_{R\vartheta}(t)=\frac{R}{2}\iint_{\Omega}\varepsilon(\varphi,\vartheta)\frac{\partial p}{\partial\vartheta}\sin\left(\frac{\vartheta}{R}\right)d\varphi d\vartheta \tag{12}$$

for $\eta=\eta(\varphi,\vartheta)$, $0\le r\le\varepsilon_T$, $0\le\varphi<2\pi\theta_1$, $0\le\theta_1<1$, $R\pi/8\le\vartheta\le R\pi/2$, $\vartheta=R\vartheta_1$.

Friction coefficients in spherical coordinates are as follows [16]:

$$\mu_{\text{sph}}=\frac{|\mathbf{e}_{\varphi}F_{R\varphi}+\mathbf{e}_{\vartheta}F_{R\vartheta}|}{C_{\text{tot}}^{(\text{sph})}}, \tag{13}$$

where $\mathbf{e}_{\varphi}$, $\mathbf{e}_{\vartheta}$ are the unit vectors in spherical (φ and ϑ) coordinate directions.

6. Numerical calculations for impulsive motion

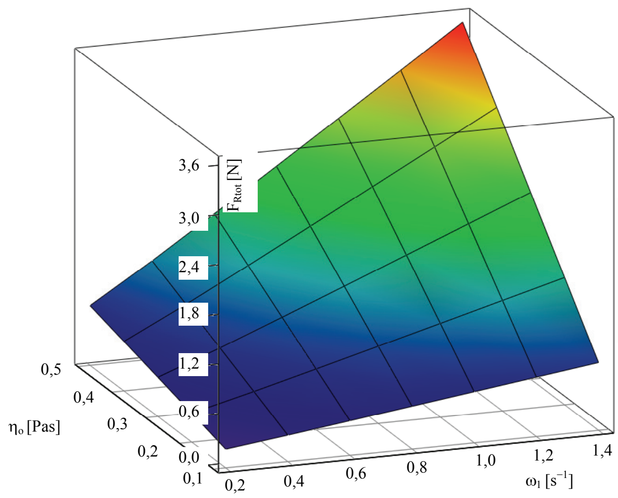
In impulsive motion, the dimensionless pressure p is determined in the lubrication region Ω by virtue of the system of equations (3)–(6), taking into account the gap height (8). The time-dependent friction forces and friction force coefficients are calculated by virtue of equations (9)–(13).

Numerical calculations for steady motion are performed in Matlab 7.2 Professional Program for radius of spherical bone head $R = 0.0265$ m, for angular velocity from $\omega = 0.2\ \text{s}^{-1}$ to $\omega = 1.4\ \text{s}^{-1}$ and characteristic dimensional time $t_0 = 0.00001$ s. The gap height (8) is taken into account at the following eccentricities of bone head: $\Delta\varepsilon_x = 4.0\ \mu\text{m}$, $\Delta\varepsilon_y = 0.5\ \mu\text{m}$, $\Delta\varepsilon_z = 3\ \mu\text{m}$. The finite differences method is applied [6].

From Dowson's experiment [1] it follows that the dynamic viscosity η of synovial fluid ranges from 0.10 Pas to 0.50 Pas. The gap height $\varepsilon_{\min}$ and gap height maximum $\varepsilon_{\max}$ are 5.77 μm and 14.7 μm, respectively. An average gap height ε equals 10.0 μm.

Moreover, we assume: the density of synovial fluid $\rho = 1010\ \text{kg/m},^3$ the average relative radial clearance $\psi \equiv \varepsilon/R = 3.774\cdot10^{-4}$.

After numerical calculations for steady motion we obtain friction force values presented in figure 5.



Rys. 5. Friction forces in human hip joint for steady motion versus angular velocity and dynamic viscosity of synovial fluid

Numerical calculations for unsteady motion are performed in Matlab 7.2 Professional Program for the radius of a spherical bone head $R = 0.0265$ m, for angular velocity in circumferential direction $\omega_1 = 1.1\ \text{s}^{-1}$, and for angular velocity in meridional direction

$\omega_3 = -0.25\ s^{-1}$, and characteristic dimensional time $t_0 = 0.00001$ s. The angular velocity representing the periodical perturbations of vibrations in synovial fluid caused by roughness is $\omega_0 = 400.0\ s^{-1}$. The gap height (8) is taken into account at the following eccentricities of bone head: $\Delta\varepsilon_x = 2.5$ μm, $\Delta\varepsilon_y = 0.5$ μm, $\Delta\varepsilon_z = 2.0$ μm. The finite differences method is applied [6].

In the calculations performed, the dynamic viscosity η of synovial fluid ranges from 0.25 Pas to 0.50 Pas. The gap height ε_{min} and gap height maximum ε_{max} are 5.80 μm and 11.50 μm, respectively. The average gap height ε equals 10.0 μm.

Moreover, we assume: the density of synovial fluid $\rho = 1010$ kg/m^3 and the average relative radial clearance $\psi \equiv \varepsilon/R = 3.77 \cdot 10^{-4}$.

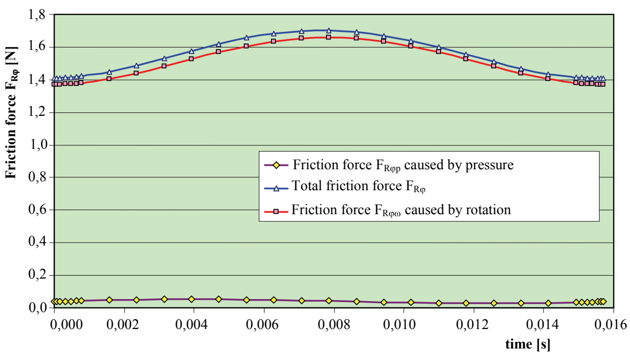


Fig. 6. Friction forces in circumferential direction of spherical human hip joint versus dimensional time at cartilage roughness perturbations caused by the pressure, rotational motion, and the sum of all influences (total friction forces)

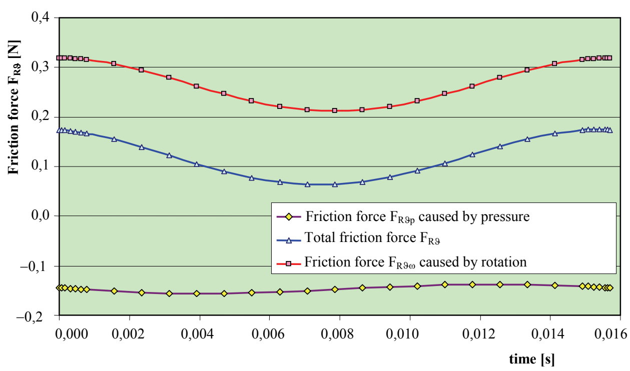


Fig. 7. Friction forces in meridional direction of spherical human hip joint versus dimensional time at cartilage roughness perturbations caused by the pressure, rotational motion, and the sum of all influences (total friction forces)

Figures 6 and 7 present the calculated values of friction forces versus time for unsteady motion caused by cartilage roughness perturbations.

By virtue of figures 6 and 7 it can easily be observed that the number of friction force changes in frequency periods are ca. 60 per second. Such perturbations are caused by real changes of the roughness on the cartilage surface.

Friction forces caused by the pressure (figures 7 and 9) have negative values, which means that these forces act in opposite directions in comparison with those of positive values.

Figures 8 and 9 present the calculated values of friction forces versus time for unsteady motion caused by the perturbations connected with human walking.

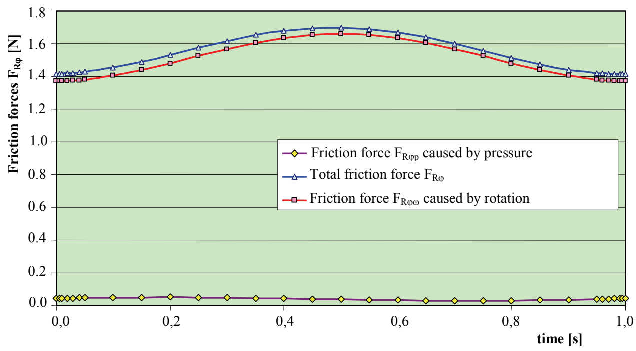


Fig. 8. Friction forces in circumferential direction of spherical human hip joint versus dimensional time at perturbations connected with human walking and caused by pressure, rotational motion, and the sum of all influences (total friction forces)

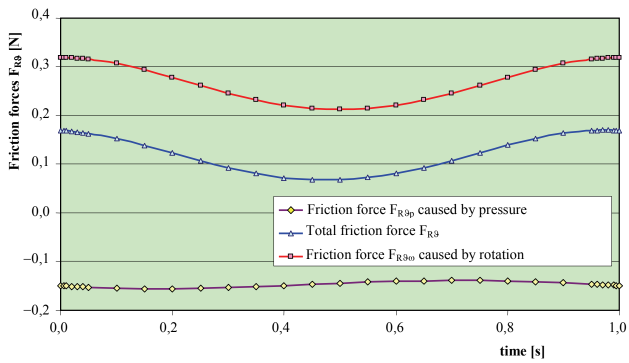


Fig. 9. Friction forces in meridional direction of spherical human hip joint versus dimensional time at perturbations connected with human walking and caused by pressure, rotational motion, and the sum of all influences (total friction forces)

7. Conclusions

1. The analytical derivation of friction forces acting in human hip joint in steady motion facilitates further numerical analyses of changes and corrections

caused by the pressure, angular velocities and magnetic induction field influencing friction forces in real hip joint.

2. The considerations presented for steady motion in the curvilinear co-ordinates make it possible to use these friction values at actual shapes of human hip joints.

3. The friction forces in steady motion increase if the velocity of synovial fluid and angular velocity of bone head increase.

4. The influences of pressure, rotational motion and the sum of all interactions on the friction forces in unsteady motion are analyzed. For each case we take into account the time perturbations of friction forces, connected with human walking and caused by the roughness of cartilage surfaces.

5. The friction force in unsteady motion of spherical bone head of human joint is greater in circumferential direction than in meridional direction.

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