Acta of Bioengineering and Biomechanics
Vol. 3, No. 2, 2001

Modelling of pulsatory flows in blood vessels

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The paper deals with the method of modelling of blood pulsatory flow in the vessels of circulatory system. The method is based upon the solution of wave equation applied in the theory of four-pole. In a graphic presentation of a vessel of distributed parameters, a model of bond graphs with new elements of DB type ise used. Basic physiological simulation experiment using bond graphs was performed. The aim of the experiment was to model the changes of pressure and volume in a left heart–aorta segment.

Key words: blood pulsatory flow, wave equation, vessel, bond graphs

Notations

a – velocity of wave propagation,
r – blood vessel's coordinate,
r_o – radius of blood vessel,
p – pressure,
υ – velocity of flow,
z – cylindrical coordinates,
A – area of blood vessel's internal section,
C – vessel compliance,
C_l – vessel's segment compliance,
E – Young's modulus,
I – inertia resistance,
I_l – vessel's segment inertial resistance,
J_o, J_2 – Bessel's functions of the first kind,
$N(s)$ – viscosity function,
Q – flow rate,
R – viscosity resistance of blood vessel,
T – time constant,
Z_l – vessel's segment impedance,
δ – wall thickness,
η – dynamic viscosity coefficient,
ν – kinematic viscosity coefficient,

ν_r – Poisson's ratio,
ρ – density,
Δz – length of vessel's segment.

1. Introduction

Difficulties in modelling of the circulatory system arise mainly due to information, calculation and measurement problems connected with, for example, representation of the arteries' elasticity, pulmonic and peripheral resistance, geometry of the circulatory system, blood viscosity and pulsatory flow. Modelling of blood flow in blood vessels requires some simplifying assumptions. On the basis of these assumptions blood vessels are treated as elastic tubes of axial-symmetric shape subjected to forces affecting the organism [5]–[9], [11], [12]. The following basic, easily accessible information is used in modelling of the circulatory system: size and mechanical properties of blood vessels, pressure changes in the left and right ventricles of the heart, pressure and flow intensity, changes in particular segments of arteries. Numerous theories concerning blood flow have been proposed, particularly those by Wamerley, McDonald [6], Rideout, Dick [7], Rudinger [13] and Skalak [14]. Digital and physical models of the circulatory system find practical applications in biomedical engineering, diagnostics and education as they are based on the laws and principles of fluid mechanics used in technical systems. Such models are used to examine the systems supporting heart operations, artificial valves and heart ventricles. They are also used to diagnose pathology, predict the operation results and to examine hemodynamic processes in blood vessels. The parameters of medical devices, for example of the apparatus supporting heart operations, can be determined by means of a digital model of the circulatory system. Most of the formulae presented in this paper can be used for rough calculation of flow intensity, pressure, velocity, flow resistance or susceptibility of vessels. The results of rough calculations are adapted to the actual values on the basis of measurements and model testing. The digital model represents physical relations by an algorithm into which actual parameters of a physical model or technical device can be given. The physical model is used in *in vitro* research as an hydraulic simulator.

2. Modelling of pulsatory blood flow

A simplified rheological model of blood represented by a Newtonian fluid described by Navier–Stokes's [2] equation is considered:

$$\rho\frac{d\upsilon}{dt} = -\operatorname{grad} p + \rho q + \eta\nabla^2\upsilon + \left(\xi + \frac{1}{3}\eta\right)\operatorname{grad}\operatorname{div}\upsilon. \tag{1}$$

Since blood is treated as a non-compressive fluid, it is assumed that div $\upsilon = 0$ and that density is constant: ρ = const. The volumetric force (component ρq) is excluded from our considerations. Further simplifications apply to velocity and pressure distributions in the radial direction in the pipe of constant cross-section. Taking all the assumptions into account we represent equation (1) as one equation of fluid motion:

$$\rho\frac{\partial \upsilon_z}{\partial t} = -\frac{\partial p}{\partial z} + \eta\left(\frac{\partial^2 \upsilon_z}{\partial r^2} + \frac{1}{r}\frac{\partial \upsilon_z}{\partial r}\right). \tag{2}$$

Next, equation (2) is transformed into general form [6], [5], [9]:

$$\frac{\partial p}{\partial z} + \frac{8\eta}{\pi r_o^4}Q + \frac{\rho}{\pi r_o^2}\frac{\partial Q}{\partial t} = 0. \tag{3}$$

Assuming a linear pressure distribution along the blood vessel:

$$\frac{\partial p}{\partial z} \cong \frac{p_2 - p_1}{\Delta z}$$

equation (3) can be rewritten as:

$$I\frac{dQ}{dt} + RQ = p_1 - p_2, \tag{4}$$

where:

$$R = 8\frac{\eta \Delta z}{\pi r_o^4},$$

$$I = \frac{\rho \Delta z}{\pi r_o^2}.$$

Equation (2) after Laplace's transformation is used to model the flow of distributed parameters [3]–[5]:

$$sQ(z,s)N(s) + \frac{A}{\rho}\frac{dp(z,s)}{dz} = 0, \tag{5}$$

where:

$$A = \pi r_o^2,$$

$$N(s) = -\frac{J_o(\gamma r)}{J_2(\gamma r)}, \tag{6}$$

$$\gamma = j\sqrt{\frac{s}{\nu}}\,. \tag{7}$$

The second equation used in modelling of pulsatory flow is the equation of continuity of non-compressive fluid flux in cylindrical coordinates:

$$\frac{\partial \upsilon_r}{\partial r} + \frac{\upsilon_r}{r} + \frac{\partial \upsilon_z}{\partial z} = 0\,. \tag{8}$$

By substitution of $\upsilon_z = Q/A$ and $\partial \upsilon_r / \partial r = \upsilon_r / r$, resulting from linear dependence $\upsilon_r = f(r)$ for boundary condition $r = r_o$, equation (8) takes the form [4]:

$$\frac{2}{r_o}\upsilon_r + \frac{1}{\pi r_o^2}\frac{\partial Q}{\partial z} = 0\,. \tag{9}$$

If we consider a blood vessel of isotropic, elastic and homogenous walls strained according to Hooke's law then for the boundary condition $r = r_o$ the change of pressure in the artery can be written as follows [2]:

$$\frac{\partial p}{\partial t} = \frac{E\delta}{(1-v_r^2)r_o^2}\,. \tag{10}$$

Since it is assumed that for a blood vessel wall Poisson's ratio $v_r = 0.5$ [12], equation (10) has the form:

$$\frac{\partial p}{\partial t} = \frac{4}{3}\frac{E\delta}{r_o^2}\upsilon_r\,. \tag{11}$$

By introducing the velocity υ_r, resulting from equation (11), into equation (9) we obtain the equation of flow continuity for length unit of the blood vessel:

$$\frac{\partial Q}{\partial z} + \frac{3\pi r_o^3}{2E\delta}\frac{\partial p}{\partial t} = 0. \tag{12}$$

Since $c = 3/2$ equation (12) takes the form [11]:

$$\frac{\partial Q}{\partial z} + c\frac{\pi r_o^3}{E\delta}\frac{\partial p}{\partial t} = 0. \tag{13}$$

By approximating:

$$\frac{\partial Q}{\partial z} \approx \frac{Q_2 - Q_1}{\Delta z}$$

and assuming:

$$\frac{\partial p}{\partial t} = \frac{dp}{dt}$$

from equation (13) we obtain the following relation:

$$C\frac{dp}{dt} = Q_1 - Q_2, \tag{14}$$

where:

$$C = c\frac{\pi r_o^3 \, \Delta z}{E\delta}. \tag{15}$$

Equation (8) after Laplace's transformations for zero initial conditions is also used to model the flow of distributed parameters [13]:

$$\frac{dQ(z,s)}{dz} + \beta s \, p(z,s) = 0, \tag{16}$$

where β is a constant coefficient connected with vessel compliance:

$$\beta = \frac{\pi r_o^3}{E\delta}. \tag{17}$$

After differentiating equation (16) by z, and taking into account the relation dp/dz determined from equation (5) we obtain wave equation:

$$\frac{d^2Q}{dz^2} - \frac{s^2}{a^2}N(s)Q = 0, \tag{18}$$

where:

$$a = \sqrt{\frac{A}{\rho\beta}}. \tag{19}$$

The general solution of equation (18) has the form:

$$Q(z,s) = c_1 e^x + c_2 e^{-x}, \tag{20}$$

where:

c_1, c_2 – constant coefficients,

x – power exponent, $x = \frac{sz}{a}\sqrt{N(s)}$.

After differentiating equation (20) by z and inserting it to formula (16) a general equation for pressure change in the vessel is obtained:

$$p(z,s) = -\frac{\rho\, a}{A}\sqrt{N(s)}\left[c_1 e^{x} - c_2 e^{-x}\right]. \tag{21}$$

By assuming boundary conditions for $z = 0$ that is $x = 0$, $p = p_1$ and $Q = Q_1$, and for $z = l$ that is $x = \frac{s\,l}{a}\sqrt{N(s)}$, $p = p_2$ and $Q = Q_2$, and by determining the integration constants c_1, c_2 from (20) and (21), and introducing hyperbolic functions cosh and sinh, the dynamic model of parameters distributed on the length l of the vessel's segment, calculated from equations (20) and (21), takes the form:

$$\begin{cases} p_2(s) = \cosh\left(\sqrt{N(s)}Ts\right)p_1(s) - Z_l\sqrt{N(s)}\sinh\left(\sqrt{N(s)}Ts\right)Q_1(s), \\ Q_2(s) = -\dfrac{1}{Z_l\sqrt{N(s)}}\sinh\left(\sqrt{N(s)}Ts\right)p_1(s) + \cosh\left(\sqrt{N(s)}Ts\right)Q_1(s), \end{cases} \tag{22}$$

where:

$$T = \sqrt{I_l C_l},$$

$$Z_l = \sqrt{\frac{I_l}{C_l}},$$

$$I_l = \frac{\rho\, l}{\pi r_o^2},$$

$$C_l = \frac{\pi r_o^3\, l}{E\delta}.$$

The matrix-vector equation of the vessel's segment finally takes the form:

$$\begin{bmatrix} p_2(s) \\ Q_2(s) \end{bmatrix} = G_s \begin{bmatrix} p_1(s) \\ Q_1(s) \end{bmatrix}, \tag{23}$$

where $\mathbf{G}_s$ is the matrix transmittance of vessel's segment:

$$\mathbf{G}_s = \begin{bmatrix} \cosh\left(\sqrt{N(s)}Ts\right) & -Z_l\sqrt{N(s)}\sinh\left(\sqrt{N(s)}Ts\right) \\ -\dfrac{1}{Z_l\sqrt{N(s)}}\sinh\left(\sqrt{N(s)}Ts\right) & \cosh\left(\sqrt{N(s)}Ts\right) \end{bmatrix} \tag{24}$$

3. Application of bond graphs

In the pulsatory systems, the elements of two input and two output parameters in the form of four-poles are considered. In the four-poles theory, the wave equation is solved by matrix-vector equation (23). Since the method of modelling the pulsatory flow by bond graph is used for each *i*th segment of blood vessel, the pressure p and the flow intensity Q on its input and output are specified:

$$\mathbf{X}_{i+1} = \mathbf{G}_i \mathbf{X}_i, \tag{25}$$

where:

$\mathbf{X}_{i+1}$ – input vector from the *i*th segment, $\mathbf{X}_{i+1} = \begin{bmatrix} p_{i+1} \\ Q_{i+1} \end{bmatrix}$,

$\mathbf{X}_i$ – input vector to the *i*th segment, $\mathbf{X}_i = \begin{bmatrix} p_i \\ Q_i \end{bmatrix}$,

$\mathbf{G}_i$ – transmittance of the *i*th vessel's segment.

For the vessel consisting of n segments the equation takes the form:

$$\mathbf{Y} = \mathbf{G}\mathbf{U}, \tag{26}$$

where:

$\mathbf{Y}$ – output vector, $\mathbf{Y} = \begin{bmatrix} p_y \\ Q_y \end{bmatrix}$,

$\mathbf{U}$ – input vector, $\mathbf{U} = \begin{bmatrix} p_u \\ Q_u \end{bmatrix}$,

$\mathbf{G}$ – vessel's matrix transmittance,

$$\mathbf{G} = \prod_{i=1}^{n} \mathbf{G}_i \mathbf{U}. \tag{27}$$

Since the dynamic model of each *i*th vessel's segment is described by different elements of bond graph (type 0, 1, TF, MTF, and also a new element DB), its $\mathbf{G}_i$ transmittance can be written as the product of elementary matrixes $\mathbf{H}_{ij}$

$$\mathbf{G}_i = \prod_j \mathbf{H}_{ij}, \tag{28}$$

where $\mathbf{H}_{ij}$ is a matrix of the *j*th element of bond graph belonging to the *i*th vessel's segment.

In figure 1, the bond graphs for the vessel's segments of concentrated and distributed parameters are compared. The matrix function $\mathbf{G_s}$ for vessel's segment of distributed parameters is represented in the bond graph by a new element DB (double bond). The element DB is represented by a double line (cf. figure 1c). In this graph, the impedances Z_1 and Z_2 are marked at the beginning and end of the blood vessel.

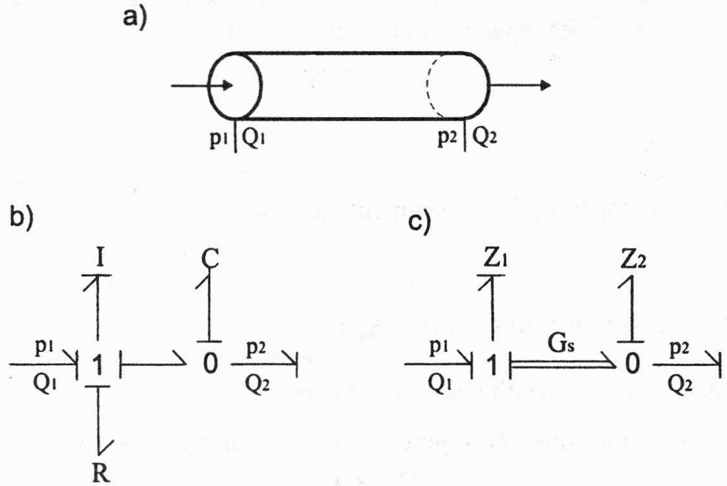

Fig. 1. Segment of blood vessel (a) bond graphs for its model of concentrated parameters (b) and distributed parameters (c)

Now we will consider a model of the vessel's segment of constant pressure distribution p and flow intensity Q, but with laminar resistance of flow R: $R = R_1 = R_2 = R_l/2$ concentrated on both ends of the vessel. In this case, after leaving out the viscosity function $N(s)$, on the basis of bond graph presented in figure 2 the equation for a model of blood vessel's segment can be written:

$$\begin{bmatrix} p_2 \\ Q_2 \end{bmatrix} = \mathbf{H}_{s3}\mathbf{H}_{s2}\mathbf{H}_{s1}\begin{bmatrix} p_1 \\ Q_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -\dfrac{1}{R_2} & 1 \end{bmatrix}\begin{bmatrix} \cosh(Ts) & -Z_l\sinh(Ts) \\ -\dfrac{1}{Z_l}\sinh(Ts) & \cosh(Ts) \end{bmatrix}\begin{bmatrix} 1 & -R_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 \\ Q_1 \end{bmatrix}$$

$$= \begin{bmatrix} G_{11}^s & G_{12}^s \\ G_{21}^s & G_{22}^s \end{bmatrix}\begin{bmatrix} p_1 \\ Q_1 \end{bmatrix}, \tag{29}$$

where:

$$G_{11}^s = \cosh(Ts),$$

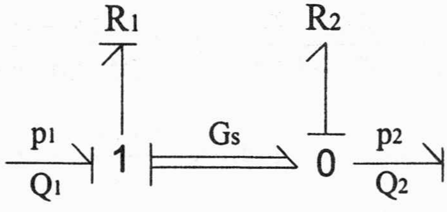

Fig. 2. Bond graph of a model of blood vessel segment of distributed parameters and local flow resistance

$$G_{12}^s = -[R_1 \cosh(Ts) + Z_l \sinh(Ts)],$$

$$G_{21}^s = -\left[\frac{1}{R_2}\cosh(Ts) + \frac{1}{Z_l}\sinh(Ts)\right],$$

$$G_{22}^s = 2\cosh(Ts) + \left(\frac{Z_l}{R_2} + \frac{R_1}{Z_l}\right)\sinh(Ts).$$

From equation (29) the impedance $Z(s)$ of the vessel's segment, which is the measure of the total resistance of blood flow, is determined. For a model of the vessel of distributed parameters the vessel impedance can be determined in the operational form $Z(n\omega)$ (where $s = j\omega$). It is developed into Fourier's series and the sum of impedances is obtained:

$$Z(n\omega) = \frac{p(n\omega)}{Q(n\omega)} = Z(0) + Z(\omega) + Z(2\omega) + \cdots + Z(n\omega). \tag{30}$$

The component $Z(0)$ is connected with the steady flow, other components are related to pulsatory flow. The vessel impedance $Z(n\omega)$ is a composite function of time and frequency. The frequency $f = \omega/2\pi$ corresponds to the frequency of heart work, while the frequencies $2f$ and $3f$, etc, are successive harmonics of the pulse wave. Since equation (29) has four unknowns the relation $R = p_2/Q_2$, in which R is flow resistance at the end of vessel, must be taken into account. Such resistance influences the wave reflection in the vessel:

- for $R > Z_l$ ($k = p_2/p_1 > 0$ and $\phi = Z_l/R < 1$) the positive reflection of wave is observed,
- for $R < Z_l$ ($k = p_2/p_1 < 0$ and $\phi = Z_l/R > 1$) the negative reflection of wave is observed (where k is reflection coefficient, and ϕ is vessel's resistance coefficient).

Thus, substituting $p_2 = RQ_2$ into formula (29) the formula for vessel's segment impedance $Z_s(s)$ is obtained:

$$Z_s(s)=\frac{p_1(s)}{Q_1(s)}=\frac{RG_{22}^s-G_{12}^s}{G_{11}^s-RG_{21}^s}=Z_l\,\frac{3\phi+(2\phi^2+1)\tanh(Ts)}{2\phi^2+\phi\tanh(Ts)}. \tag{31}$$

From formula (31) the spectral impedance $\frac{Z_s(j\omega)}{Z_l}$ is obtained. From the spectral impedance the modulus $\left|\frac{Z_s(j\omega)}{Z_l}\right|$ and the argument $\arg\left(\frac{Z_s(j\omega)}{Z_l}\right)$ are derived. They are used to determine the resonant characteristic of blood vessel for different frequencies of heart work.

4. A physiological simulation benchmark experiment using bond graphs

This paper presents a classical model of the human blood vessels, which are part of the circulatory system, a good example of a non-linear continuous system. The model components used to simulate human circulatory system are best described in terms of physiological elements that they represent. Some of the systems that constitute the human body are in direct contact with the cardiovascular system, while the others are not. The circulatory (i.e. cardiovascular) system can be divided into three distinct parts: pulmonary (small) circulation passing blood to the lungs, systemic (large) circulation providing the rest of the body with blood and the heart with its own vasculature (coronary circulation). Heart plays a role of the double pump ("right heart" and "left heart"). One pump collects the blood (through venae cavae) from the periphery and passes it into the lungs to become there oxygenated ("right heart"). The other one collects the oxygen-rich blood from the lungs and passes it back to the periphery through aorta ("left heart"). The systemic circulatory system was divided into four subsystems: arms, head, trunk and legs. The scheme of the model of the circulatory system is shown in figure 3. Each of the model elements (head, lungs, right heart, left heart, vena cava, arms, aorta, legs, trunk) has four basic input parameters used for describing its characteristics. Two of these parameters (R – resistance in mm Hg/cm^3/s, C – compliance in cm^3/mm Hg) are key elements of the basic blood flow components. Each of the model elements also has basic output parameters. Three of these are characteristic of blood flow (p – pressure in mm Hg, Q – flow rate in cm^3/s, V – volume in cm^3). Parameter Control Centre (PCC) assigns values to changing parameters in the simulation model presented in the table.

Blood volume corresponds to the V_n term for each of the n subsystem

$$V_n-V_{no}=\int_0^t (Q_n-Q_{n+1})dt\,. \tag{32}$$

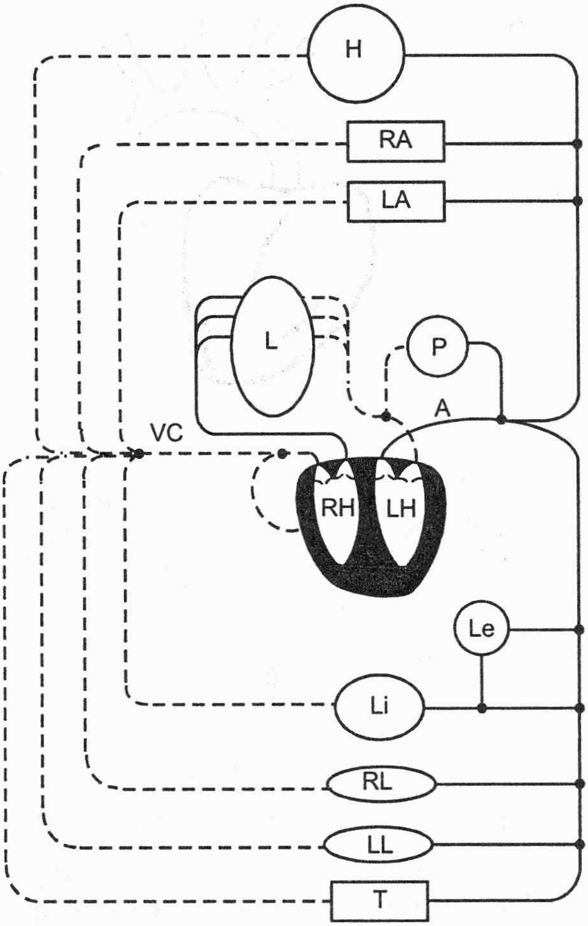

Fig. 3. Scheme of the model of the human circulatory system: A – aorta, H – head, L – lungs, Li – liver, Le – lien, LA – left arm, LH – left heart, LL – left leg, P – pericardium, RA – right arm, RH – right heart, RL – right leg, T – trunk, VC – vena cava

Output pressure corresponds to the p_n term for each of the n subsystems

$$p_n = \frac{1}{C_n} \Delta V_n . \tag{33}$$

Simulation model is based on the same assumptions as all basic classic models, i.e. blood without mass, blood flow modelled as a Newtonian fluid, organs and blood vessels having linear compliance, heart valves that close immediately, lumped body systems. The application of bond graphs in modelling of pulsatory blood flow in LH–A (left heart–aorta) subsystem is presented in figure 4. LH model favours the left ventricle, ignoring the lesser flow effects in the left atrium. The mitral and aortic valves

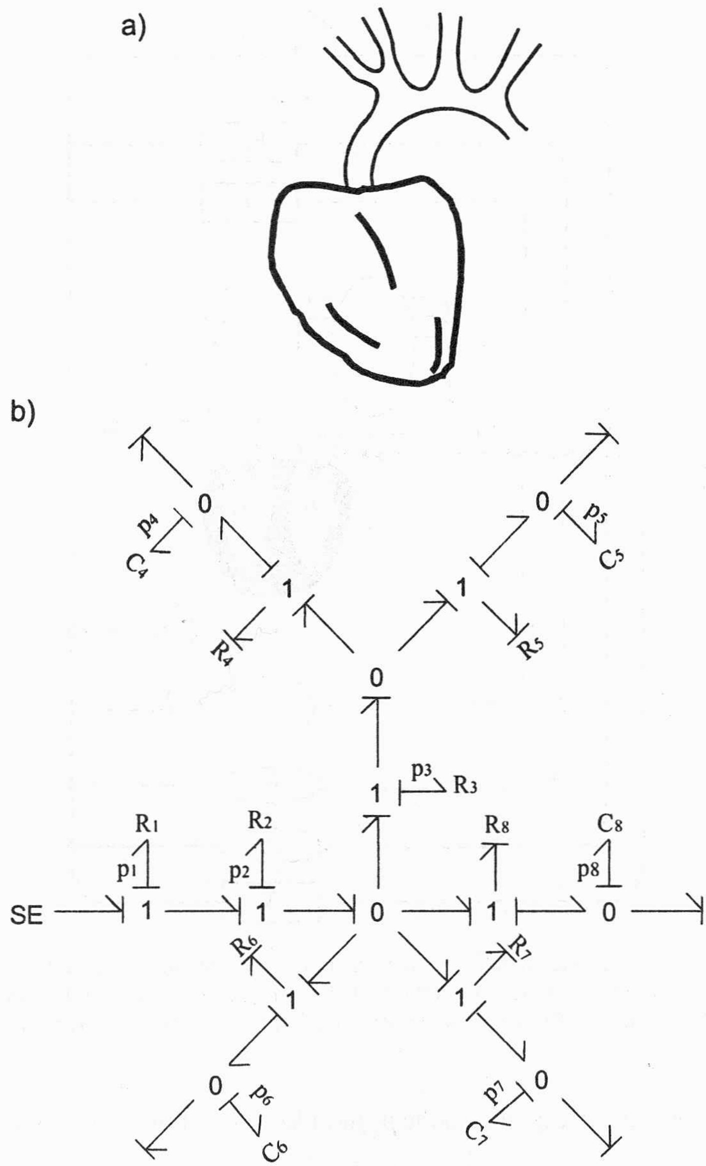

Fig. 4. The LH–A (left heart–aorta) subsystem (a) and its bond graphs (b)

stop the flow to and from the left ventricle, respectively, when the pressure difference across the valve becomes negative (i.e. they are unidirectional valves that prevent backflow; mitral valve controls blood flow from the left atrium into the left ventricle; aortic valve controls blood flow from the left ventricle into the aorta; ventricles – two chambers at the bottom of the heart that form a pointed base, main pumping chambers). Blood temperature is assumed to remain unchanged in the heart. Input resis-

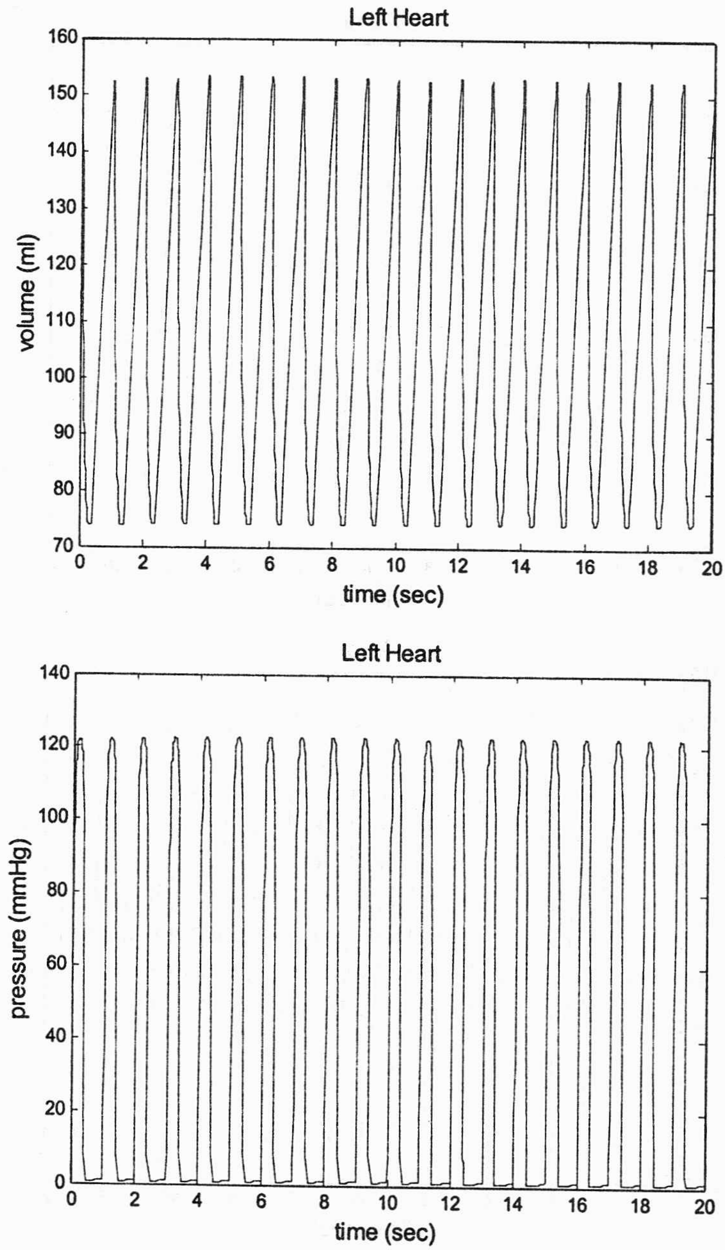

Fig. 5. Plots of volume V_l and pressure p_l data for the left heart

tance of the aorta is equivalent to the output resistance of the right ventricle. Output resistance of the aorta is equivalent to the combined input resistance of the arms, head, trunk, and legs. The simulation of the circulatory system was conducted by means of a Simulink and 20sim Performance Tools adapted to the structure of the bond graphs.

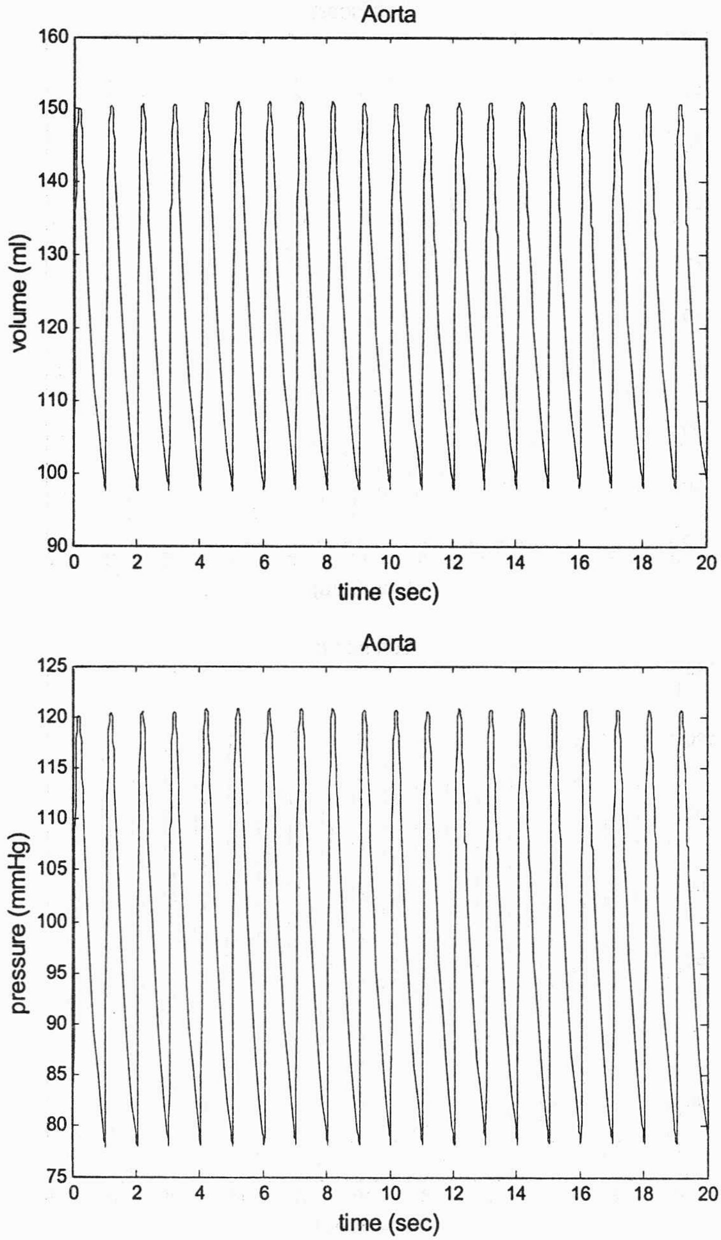

Fig. 6. Plots of volume V_8 and pressure p_8 data for the aorta

The plots of pressure and volume data for the left heart are presented in figure 5, and plots of pressure and volume data for the aorta are presented in figure 6. The time range covered is 20 sec.

Table. Changing parameters in the simulation model [15]

Elements of circulatory system	R_1 [mm Hg/cm³/s]	R_2 [mm Hg/cm³/s]	C [cm³/mm Hg]	V_0 [cm³]
Right heart	0.011	0.0128	75	150
Lung	0	0.1429	7.519	120
Left heart	0.0125	0.0588	80	150
Aorta	0	0	1.25	100
Arms	10	5.15	4.25	280
Head	5	2.58	1.21	80
Trunk	1.42	0.67	34	2250
Legs	5	2.58	11.1	730
Vena cava	0	0	250	500

5. Conclusions

The paper deals with modelling of pulsatory flow in blood vessels on the basis of the laws and principles of fluid mechanics used in technical systems. The analogies and principles of hydrodynamic similarities are applied. In modelling of pulsatory flow, the matrix-vector equation is used as the solution of the wave equation used in the four-pole theory. For graphic presentation of pulsatory flow, bond graph with a new element of DB (double bond) type is applied. Such a bond graph enables modelling of dynamics with regard to wave phenomena occurring in blood vessels. The described method of modelling of pulsatory flow in cardiovascular system enables one to determine the transmittance and impedance of a blood vessel segment and provides a basis for determination of resonant characteristics for different frequencies of heart work. Theoretical frequency characteristics are compared with actual characteristics obtained during *in vitro* research on a hydraulic simulator.

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